Quiver Representations, Quiver Varieties and Combinatorics
Bologna, May 2023

Quiver Representations in Topological Data Analysis (TDA)

Steve Oudot

Inria

References

- 1-parameter persistence theory:

  O. (2015): Persistence Theory: from Quiver Representations to Data Analysis.

- Multi-parameter persistence theory:

  Botnan, Lesnick (2022): An Introduction to Multi-Parameter Persistence.

- Algorithmic aspects:

  Dey, Wang (2022): Computational Topology for Data Analysis.

- Statistical aspects:

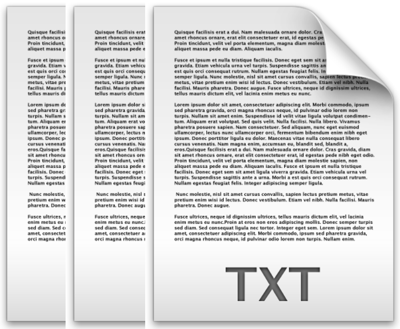
  Chazal, Michel (2021): An Introduction to Topological Data Analysis.

- Connection to Machine Learning:

  Hensel, Moor, Rieck (2021): A Survey of Topological Machine Learning Methods.

- Software: *Gudhi*, *PHAT*, *Ripser*, *Eirene*, *Persistable*, ...

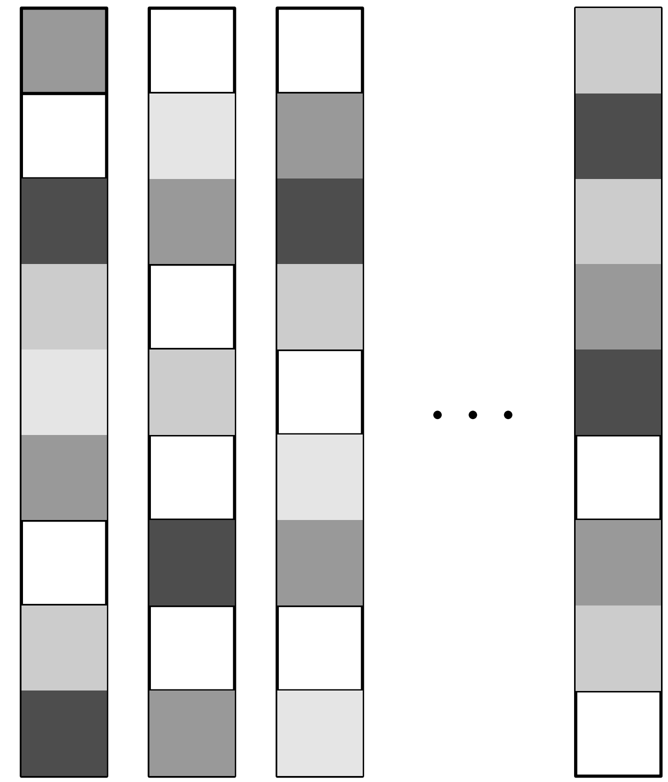
Data featurization

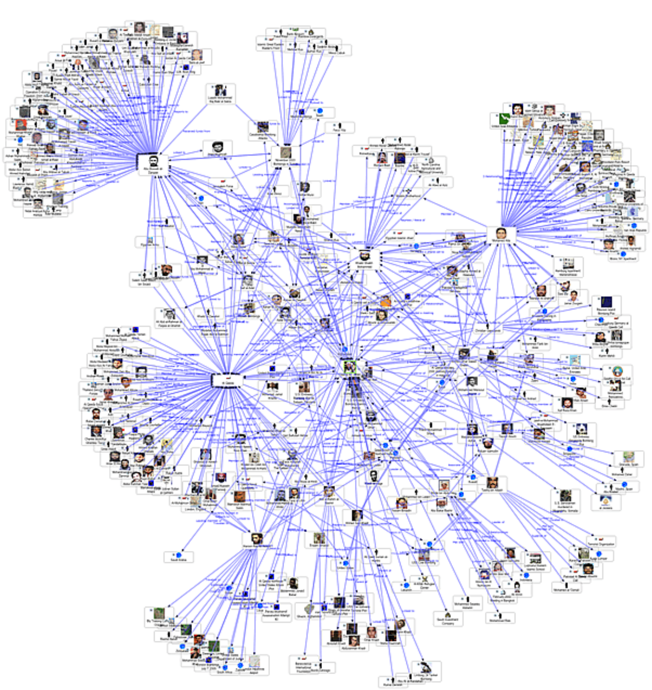


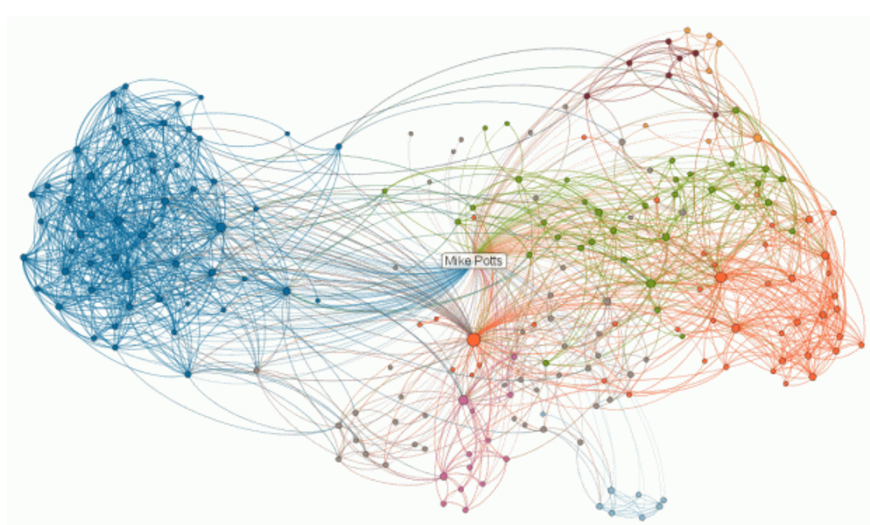
Data

Features $\in \mathbb{R}^n$

(feature design or learning)







bag of words, word2vec

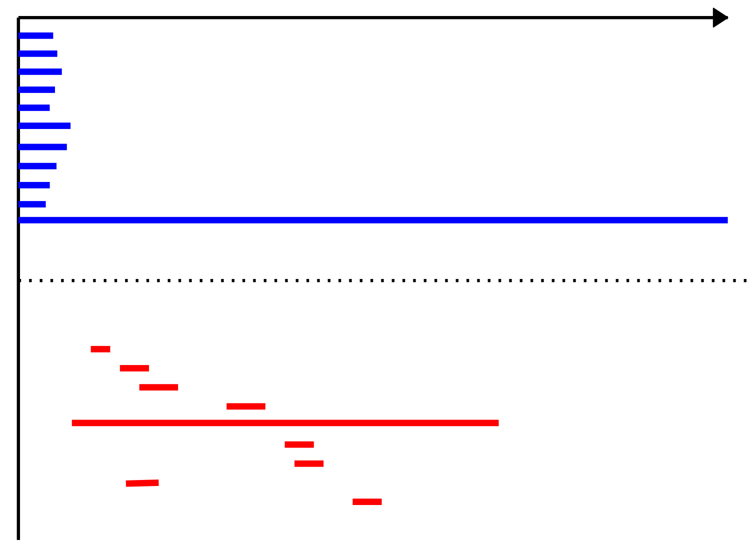
shape contexts, heat kernels

node2vec, Laplacian fact., rand. walks

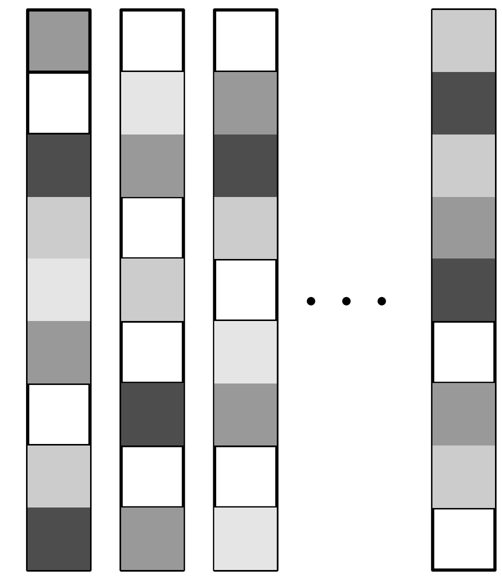
dim. reduction, auto-encoders, etc.

Topological Data Analysis pipeline

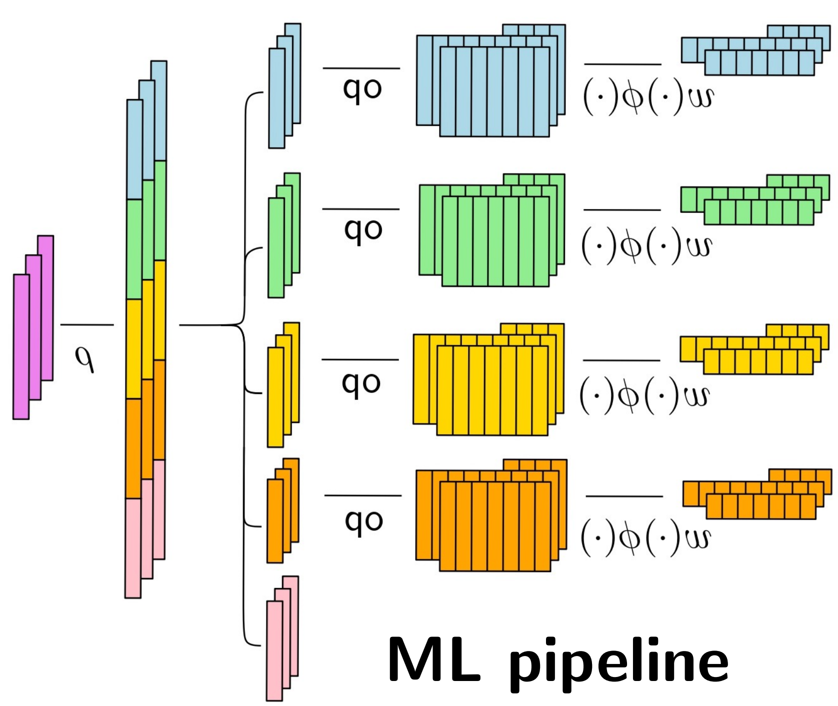
data



invariant (barcode)



features (vectors)



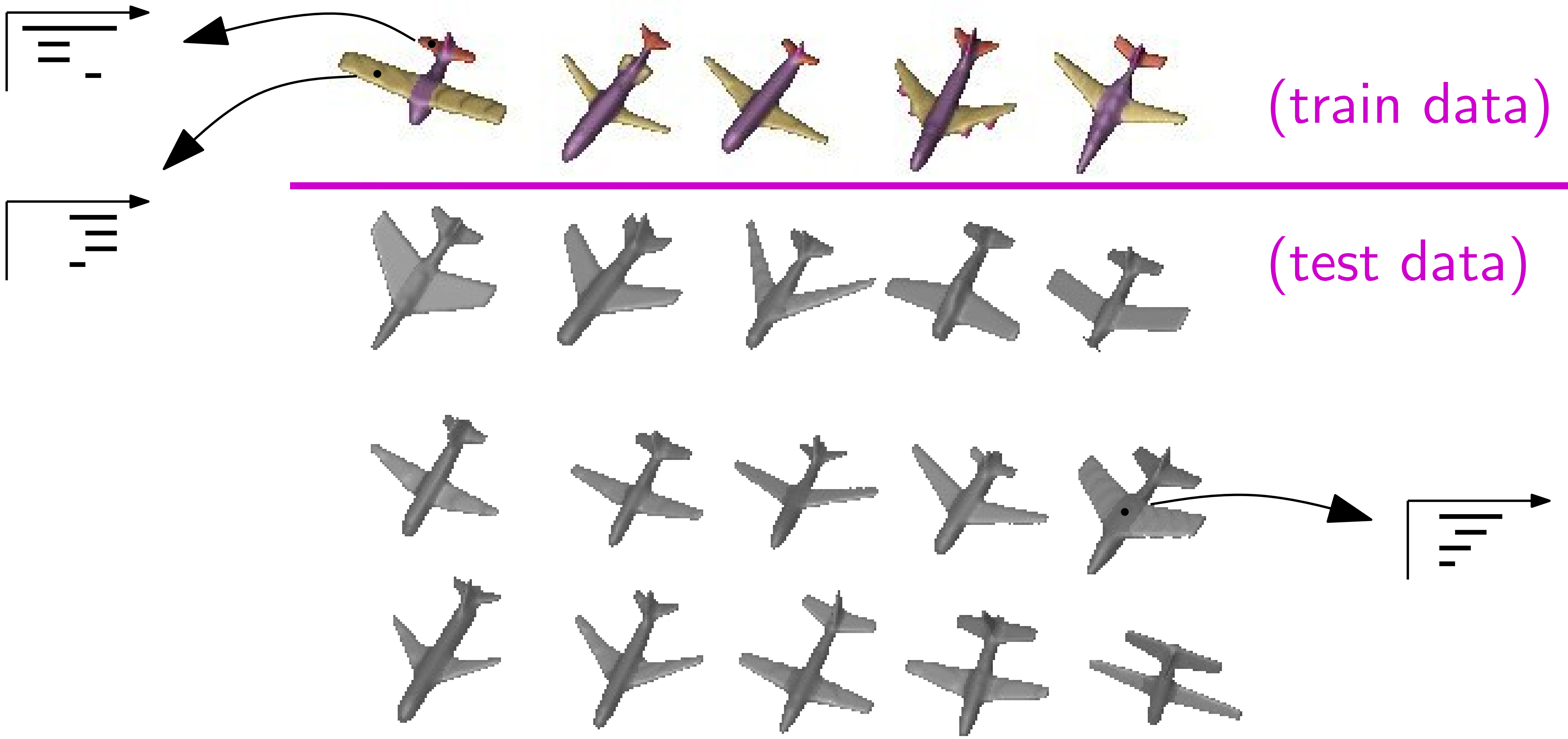
ML pipeline
(e.g. neural net.)

Example of application: shape segmentation

Goal: segment 3d shapes based on examples

Approach:

- train a predictor on **barcodes** extracted from the training shapes
- apply the predictor to **barcodes** extracted from the query shape



[Carrière, O., Ovsjanikov]

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Error rates (%):

	TDA features	geom/stat features	TDA + geom/stat features
Human	26.0	21.3	**11.3**
Airplane	27.4	18.7	**9.3**
Ant	7.7	9.7	**1.5**
FourLeg	27.0	25.6	**15.8**
Octopus	14.8	5.5	**3.4**
Bird	28.0	24.8	**13.5**
Fish	20.4	20.9	**7.7**

[Carrière, O., Ovsjanikov]

Topological Data Analysis pipeline (again)

data

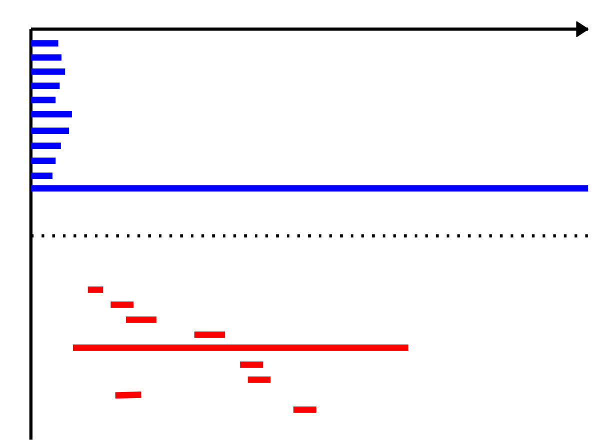
filtration $F : P \subseteq \mathbb{R}^d \to \mathsf{Top}$

(sublevel sets of distance to data)

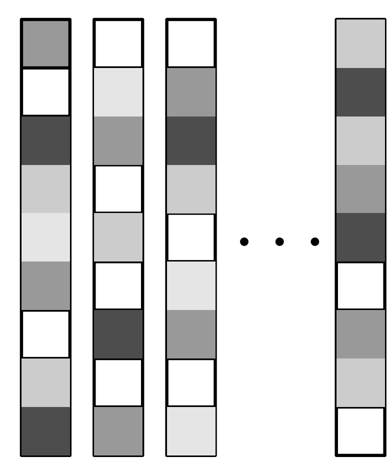
persistence module

$M = HF : P \to \mathsf{vect}_{\boldsymbol{k}}$

(singular homology)



invariant $(\mathrm{Bar}\, M)$



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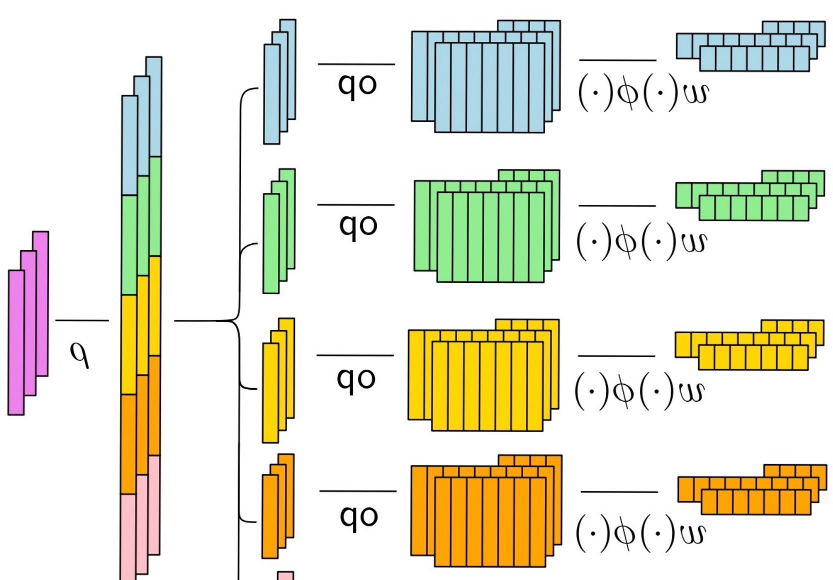

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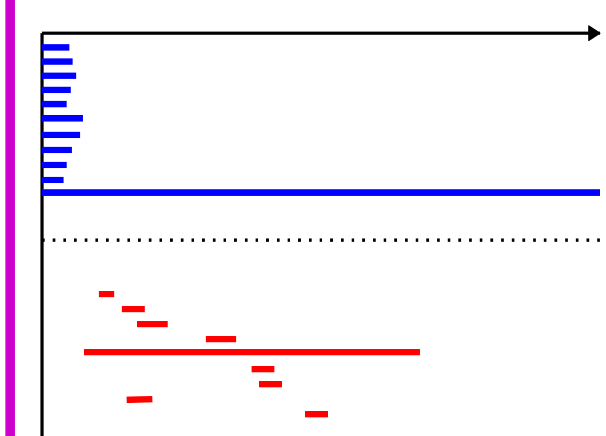

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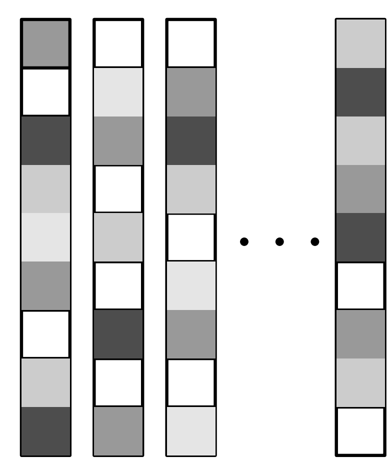

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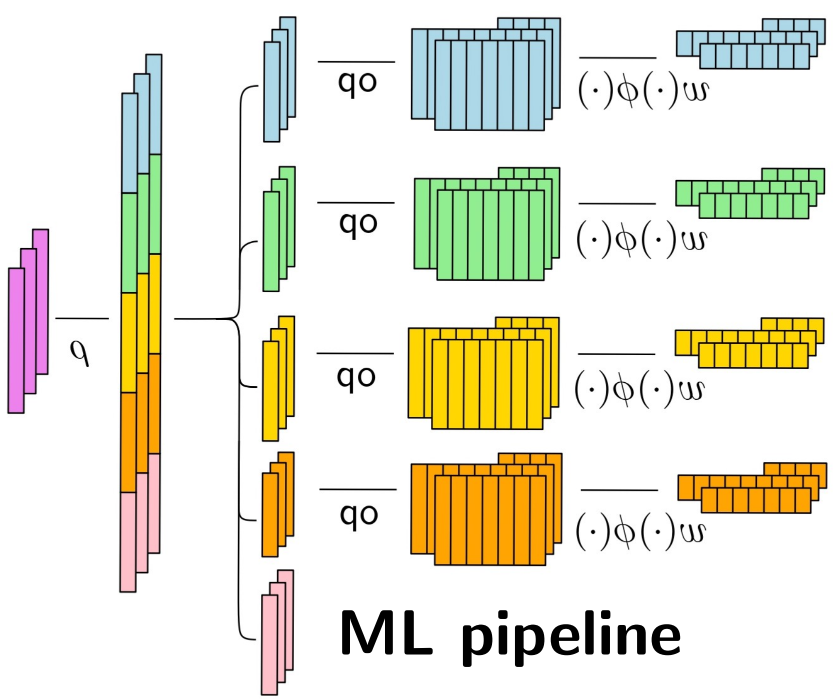

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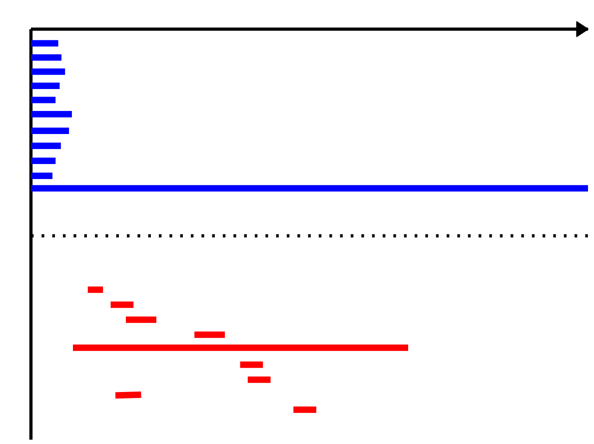

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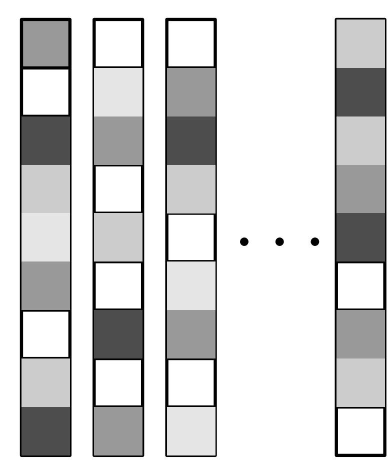

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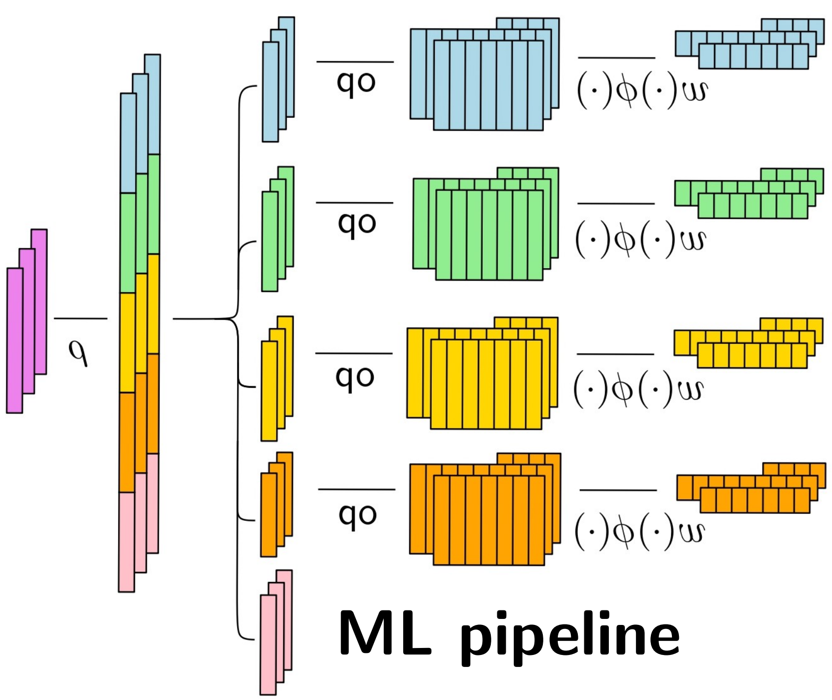

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Filtrations

$(P, \leq)$ a poset (usually $(\mathbb{Z}^d, \leq_{\prod})$ or $(\mathbb{R}^d, \leq_{\prod})$)

Filtration: functor $(P, \leq) \to$ Top

- typically, $F(t) := f^{-1}((-\infty, t])$ for some P-valued function f
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Example: offsets filtration of $X \subseteq \mathbb{R}^n$:

$$f : \left| \begin{array}{l} \mathbb{R}^n \to P = \mathbb{R} \\ y \mapsto \min_{x \in X} \|y - x\|_2 \end{array} \right.$$

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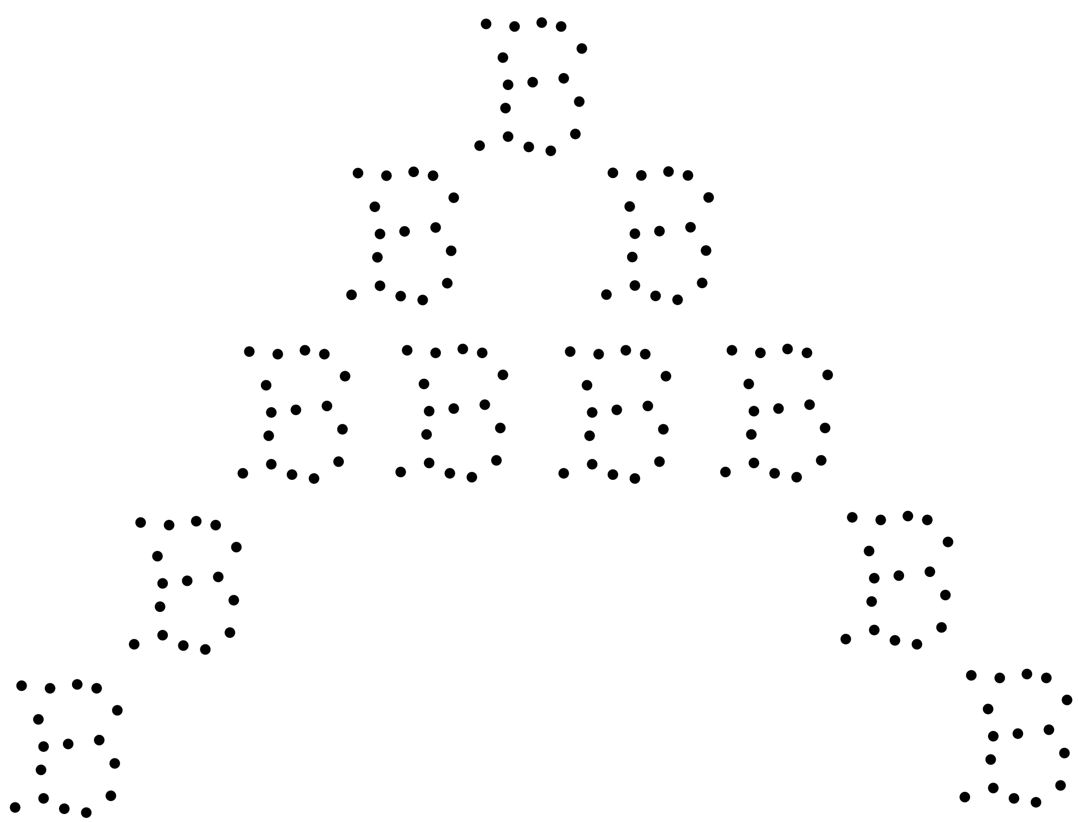

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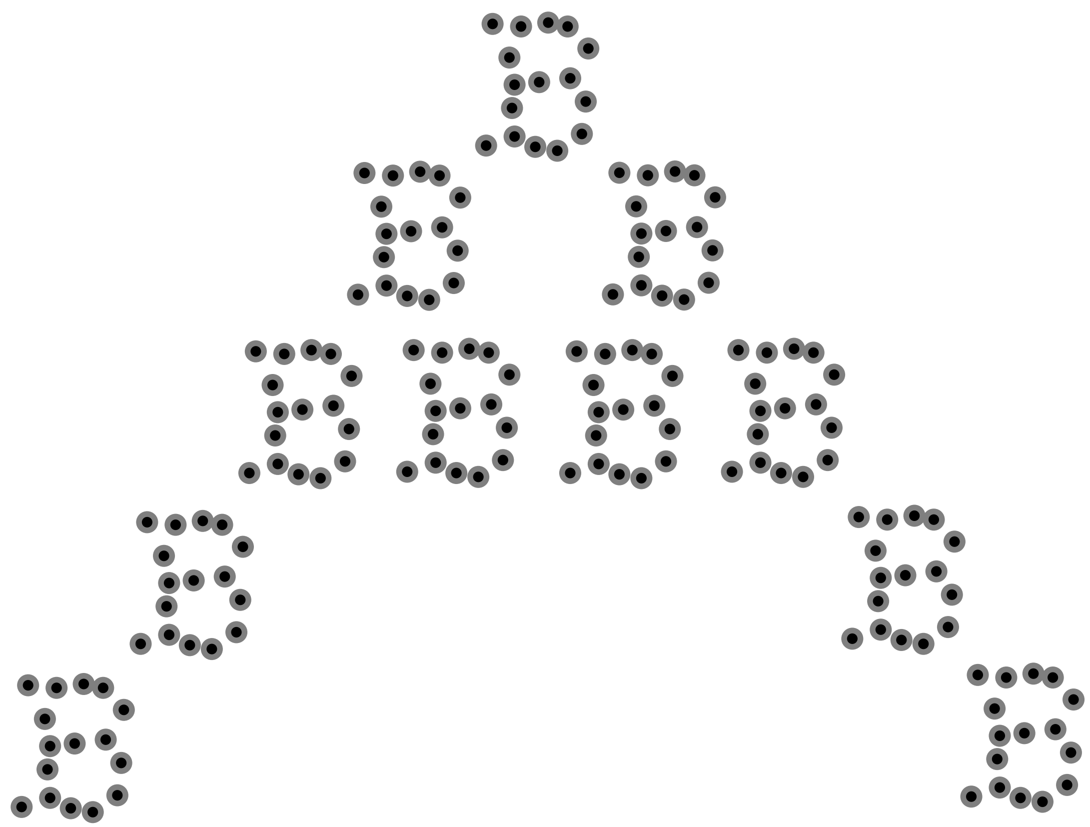

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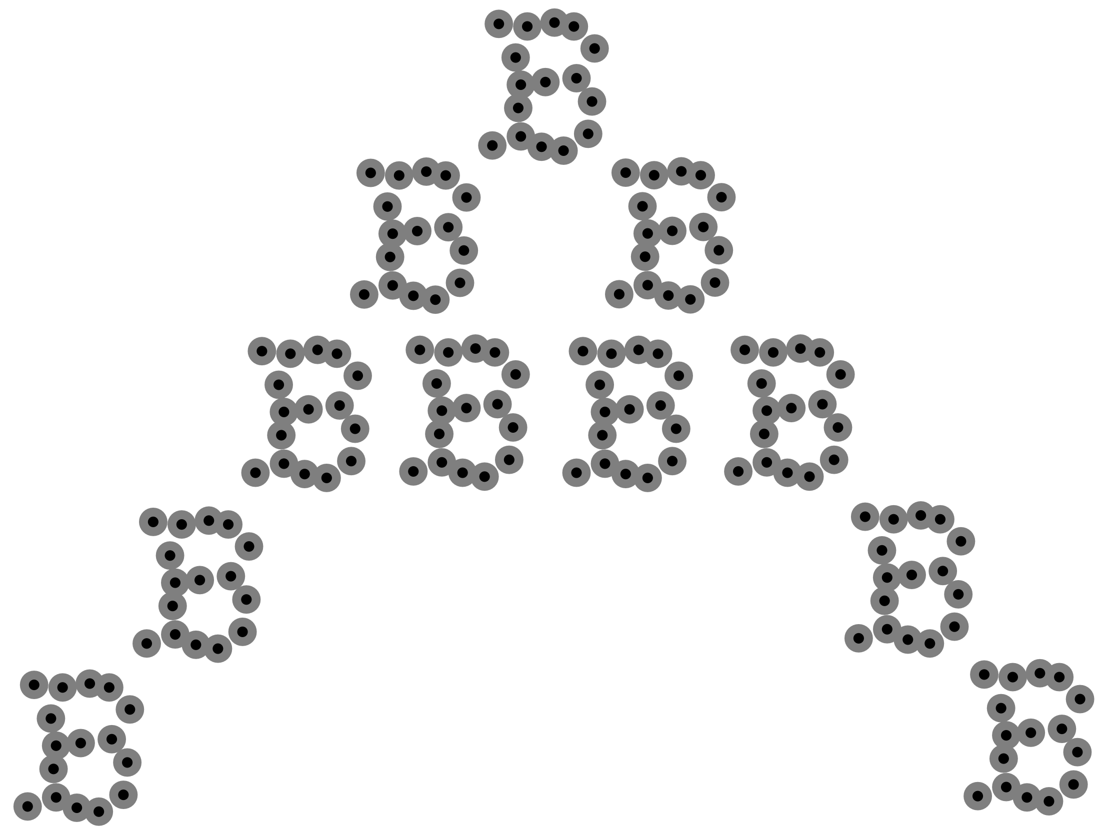

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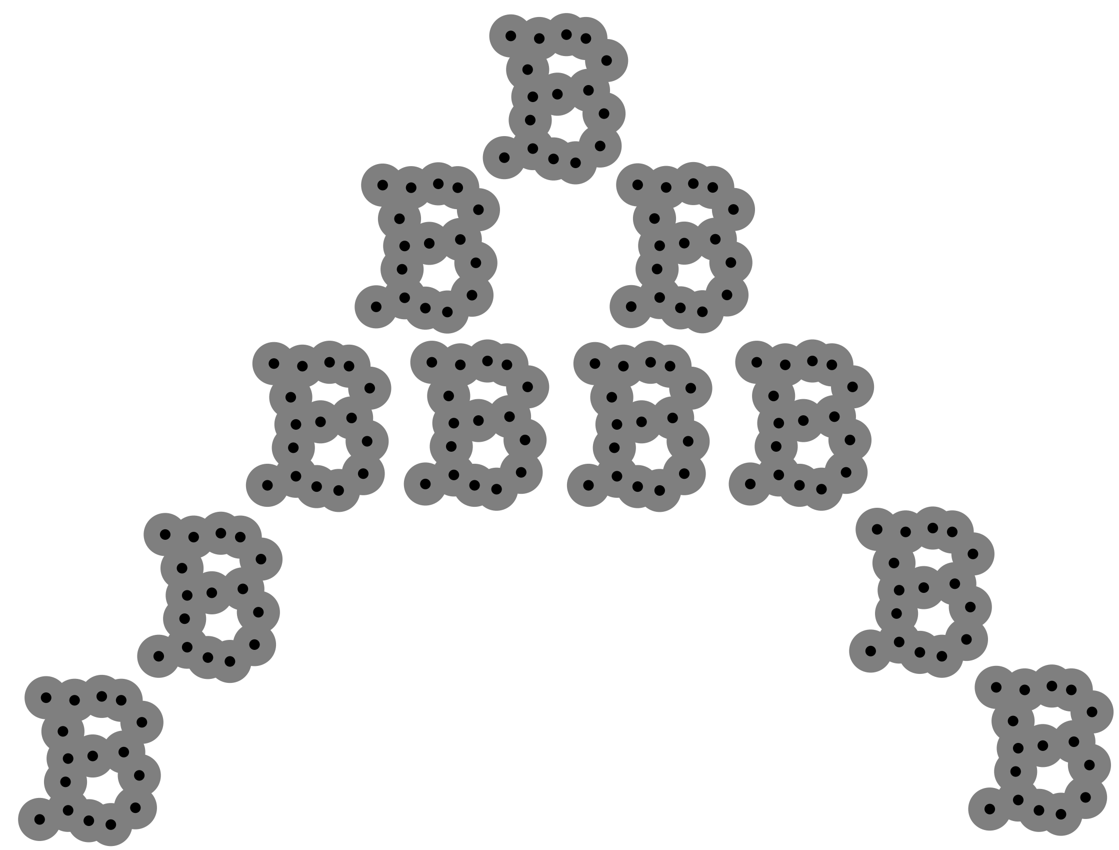

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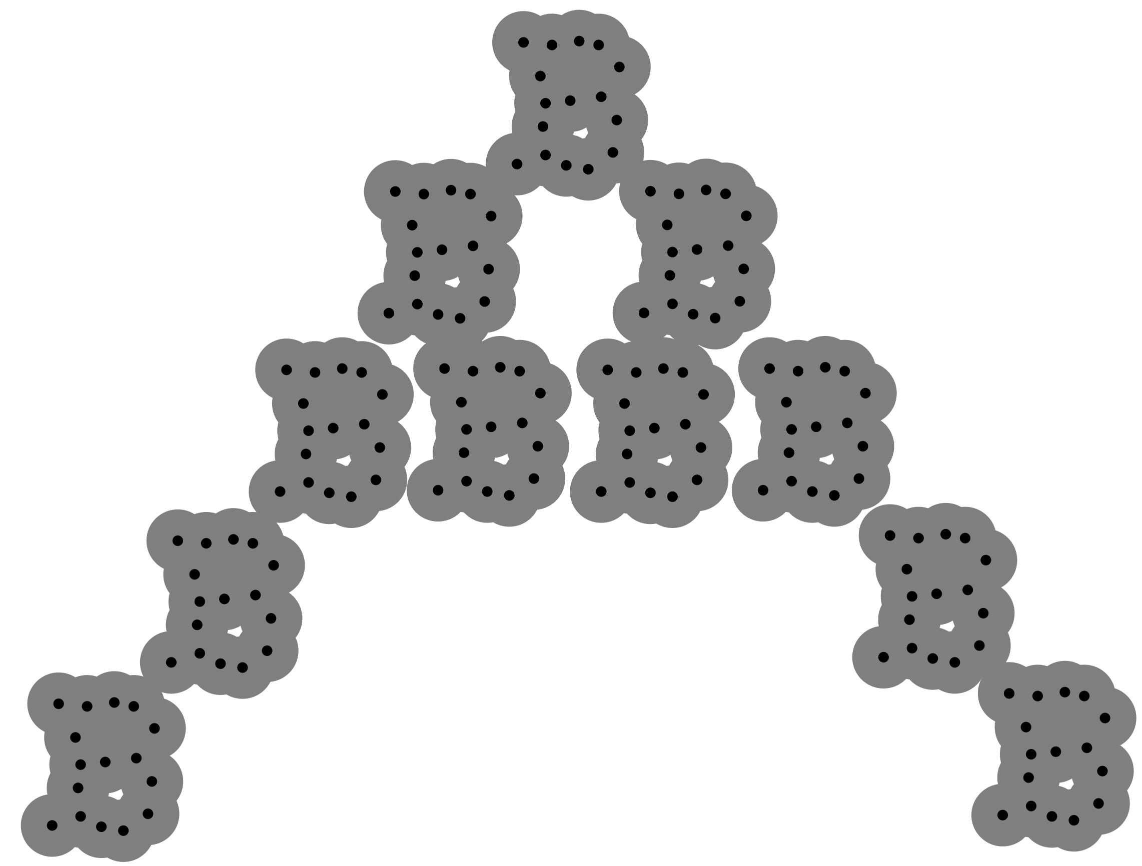

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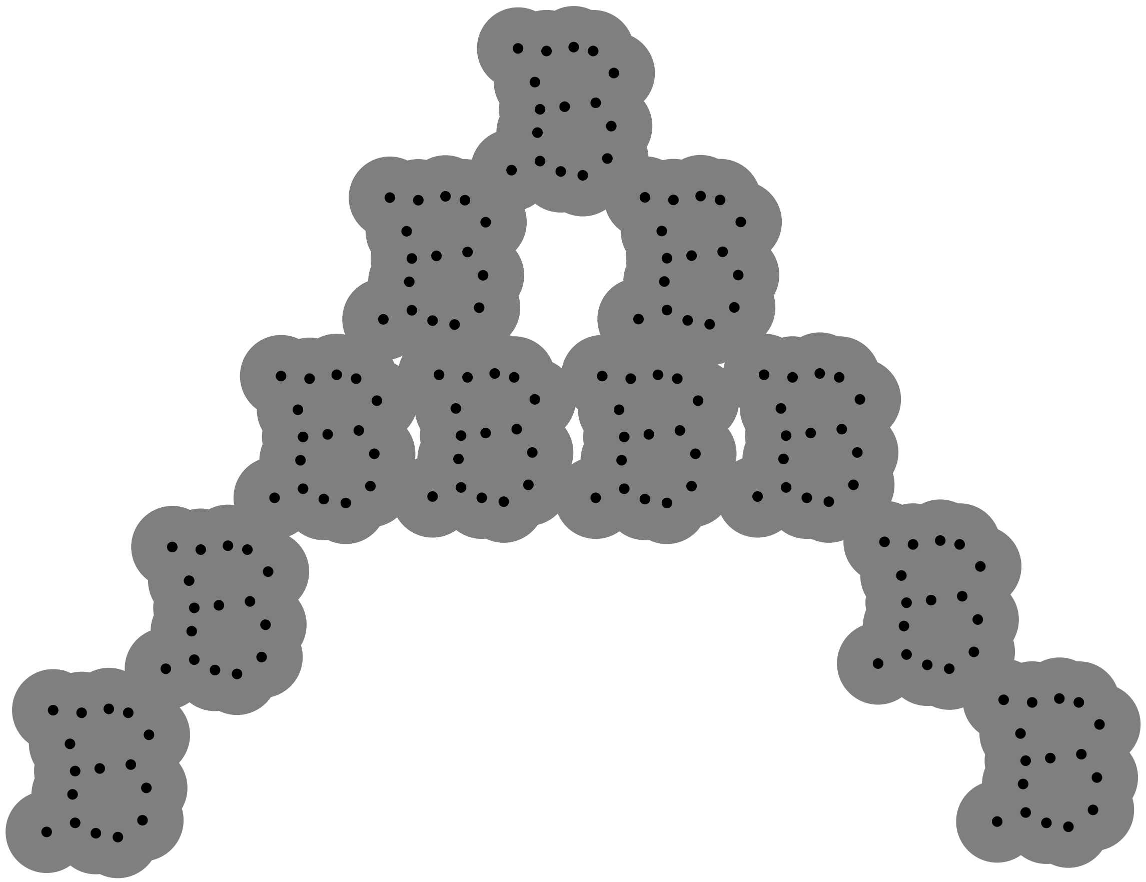

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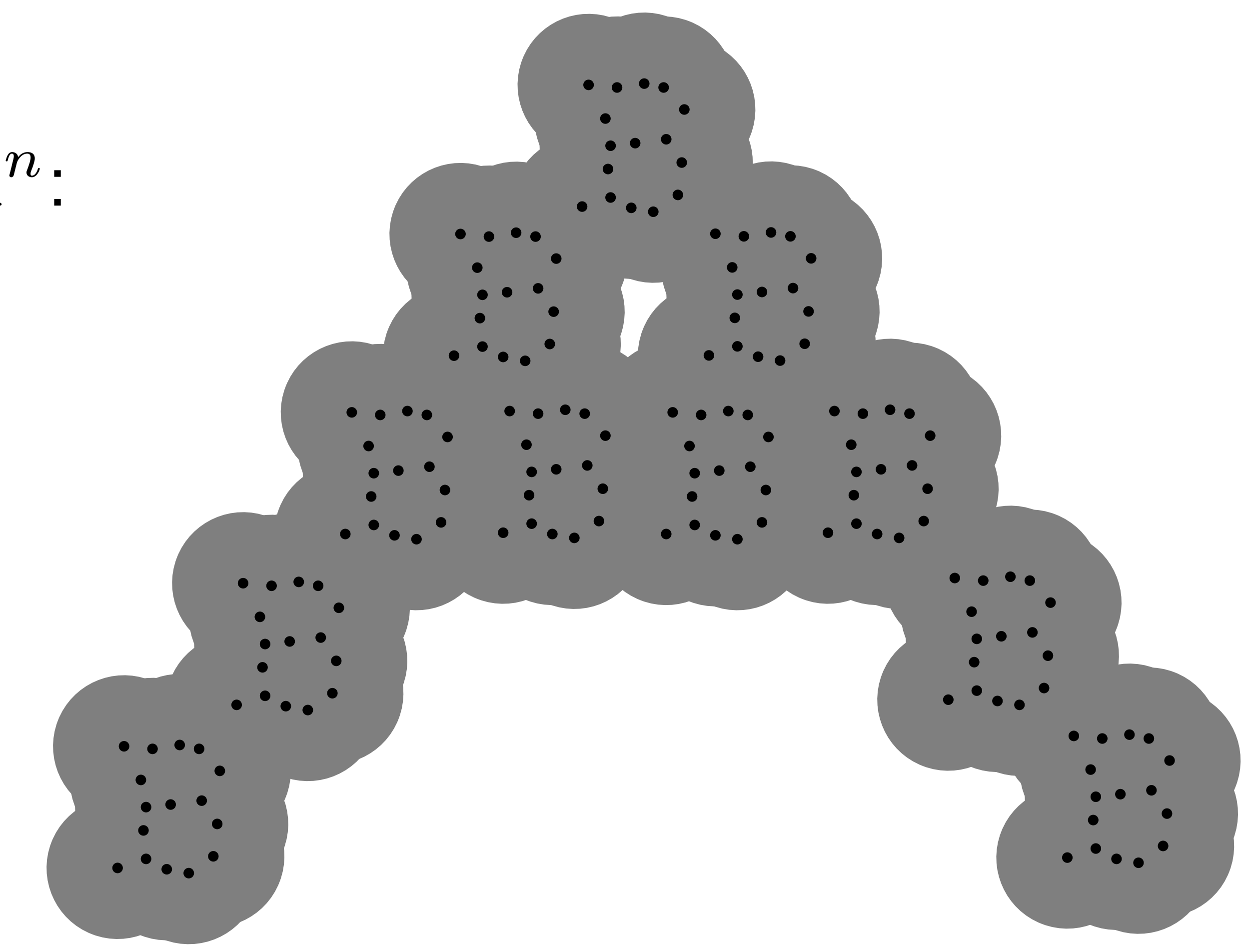

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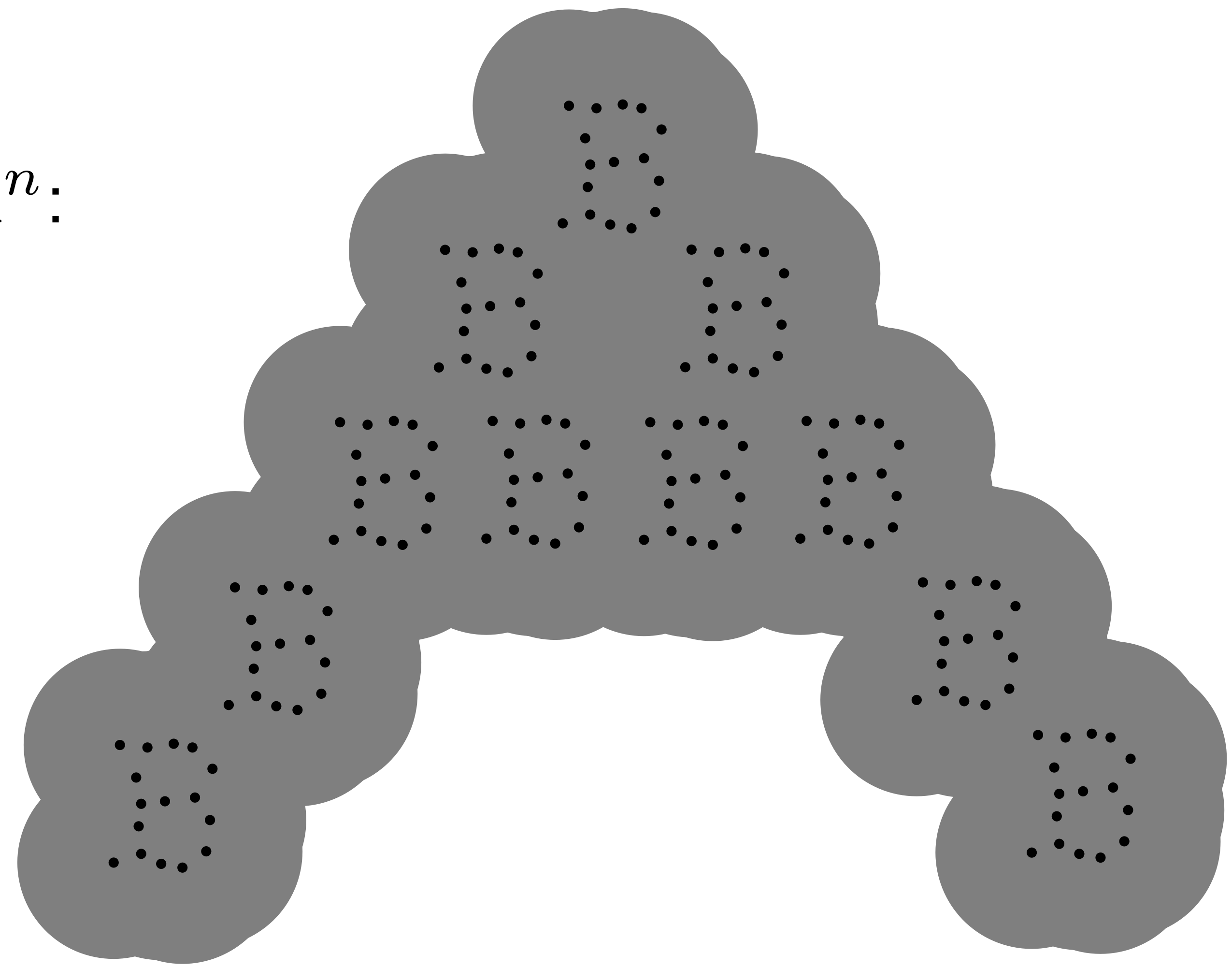

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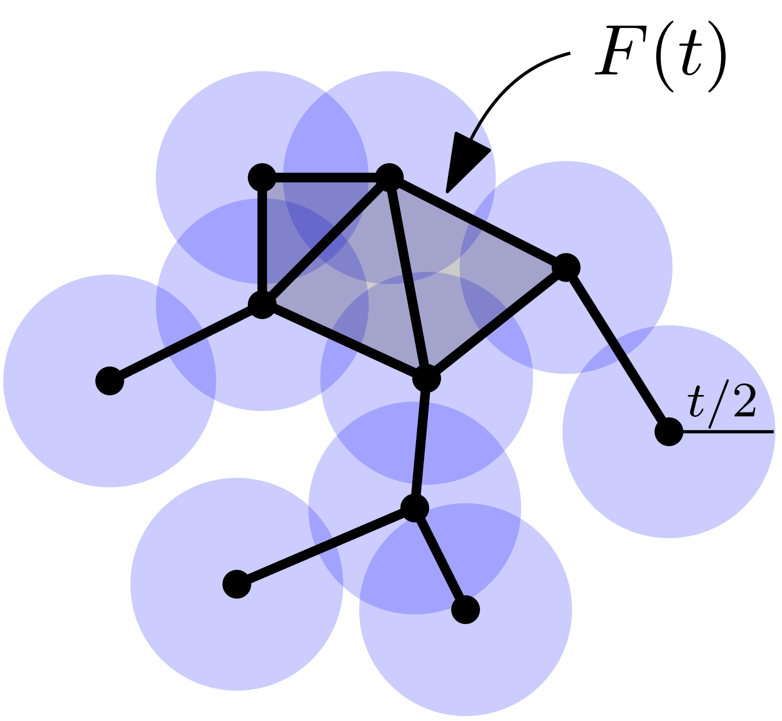
- for computational purposes, take $F \colon P \to$ Simp

Example: Vietoris-Rips filtration of $X \subseteq \mathbb{R}^n$:

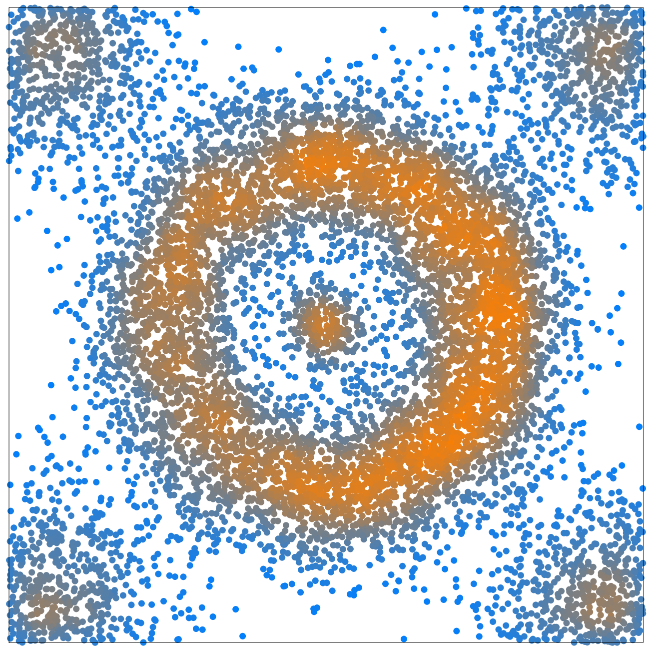
$$f : \left| \begin{array}{l} 2^X \to P = \mathbb{R} \\ \{x_0, \cdots, x_k\} \mapsto \max_{0 \leq i < j \leq k} \|x_i - x_j\|_2 \end{array} \right.$$

$F(t) =$ flag complex of intersection graph

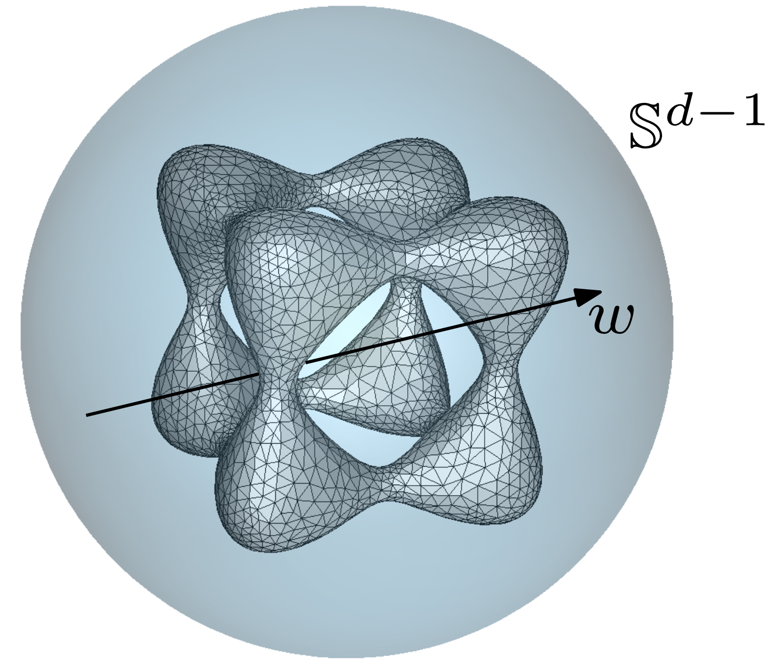
of $\bigcup_{x \in X} B(x, t/2)$



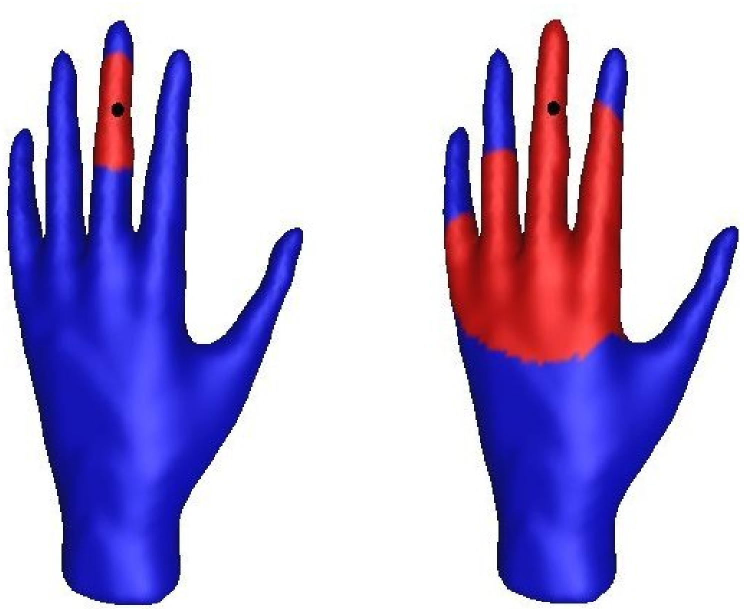
Other examples (any combination of the following)



density estimators



projections

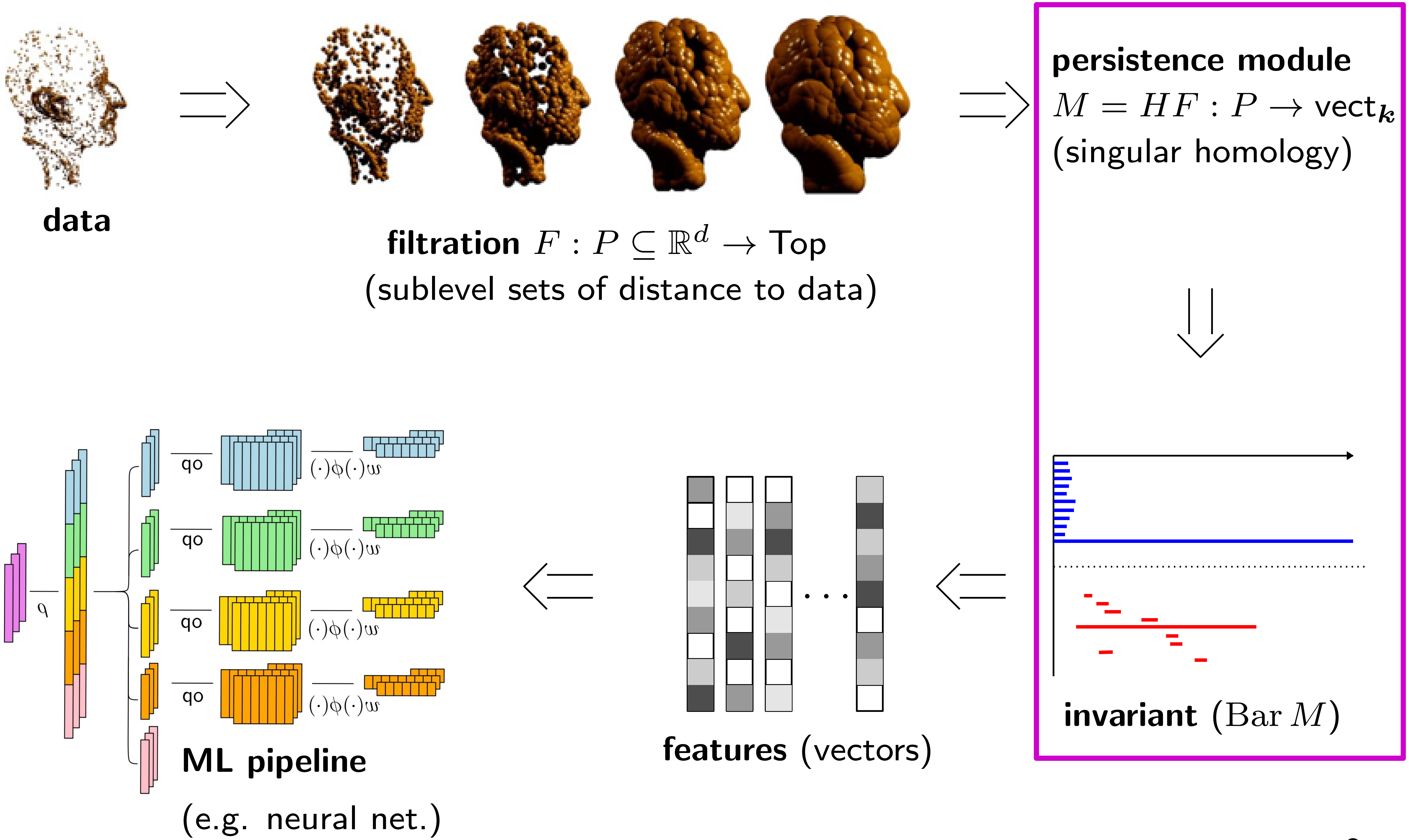


single-source distances

others:

- non-linear projections
- curvature measures
- PDE solutions (heat, wave)
- etc.

Topological Data Analysis pipeline (again)



Persistence modules

$(P, \leq)$ a poset (usually $(\mathbb{Z}^d, \leq_\prod)$ or $(\mathbb{R}^d, \leq_\prod)$), $\boldsymbol{k}$ a field

Persistence module: functor $(P, \leq) \to \mathsf{vect}_{\boldsymbol{k}}$ (pointwise finite-dimensional, or pfd)

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Persistence module: functor $(P, \leq) \to \mathrm{vect}_{\boldsymbol{k}}$ (pointwise finite-dimensional, or pfd)

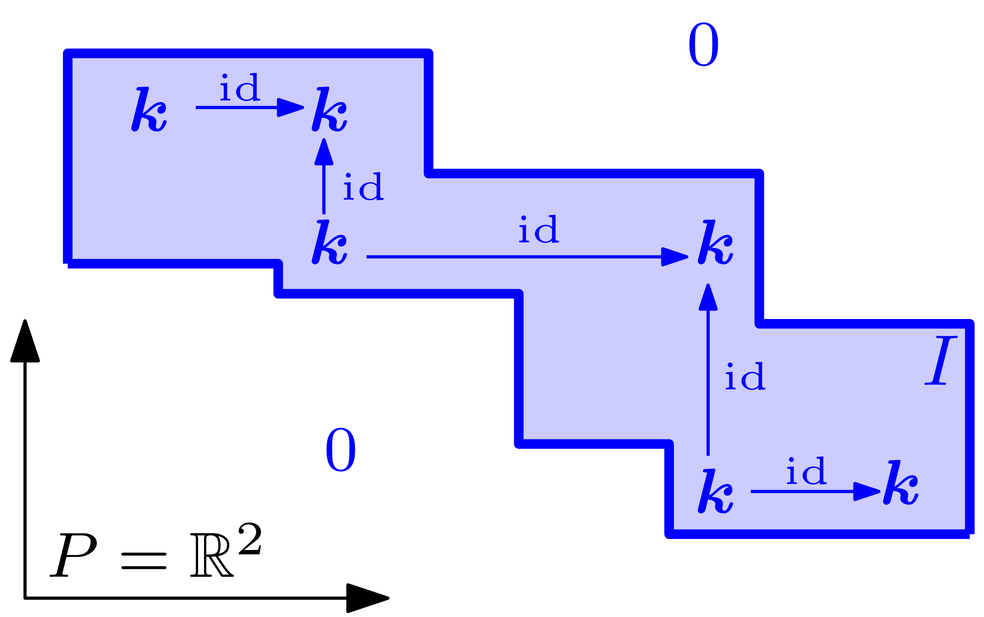
Interval: $I \subseteq P$ that is:

- convex ($s, t \in I \implies u \in I \; \forall \; s \leq u \leq t$)
- connected ($s, t \in I \implies \exists \{u_i\}_{i=0}^r \subseteq I$ s.t. $s = u_0 \leq u_1 \geq \cdots \geq u_r = t$)

Interval module: indicator module $\boldsymbol{k}_I$ of an interval $I \subseteq P$

$$\boldsymbol{k}_I(t) = \begin{cases} \boldsymbol{k} \text{ if } t \in I \\ 0 \text{ otherwise} \end{cases}$$

$$\boldsymbol{k}_I(s \leq t) = \begin{cases} \mathrm{id}_{\boldsymbol{k}} \text{ if } s, t \in I \\ 0 \text{ otherwise} \end{cases}$$



note: $\mathrm{End}(\boldsymbol{k}_I) \simeq \boldsymbol{k}$

Persistence modules

Interval modules are **described by their support**:

- complete geometric descriptor
- efficient to encode (small / simple dictionary)
- readily interpretable (for data exploration)
- easy to vectorize (for Machine Learning)
- enjoy stability properties (for statistics)

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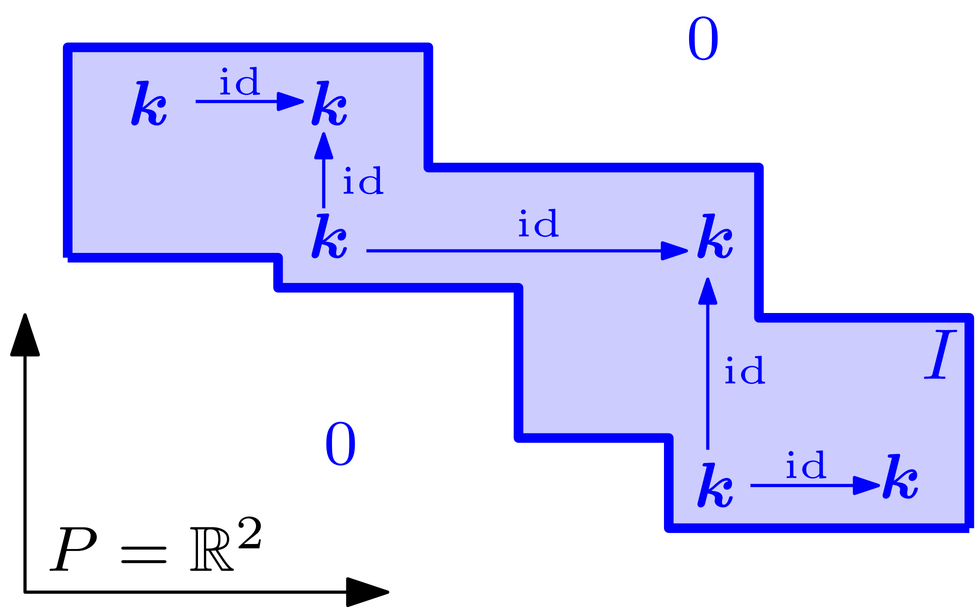

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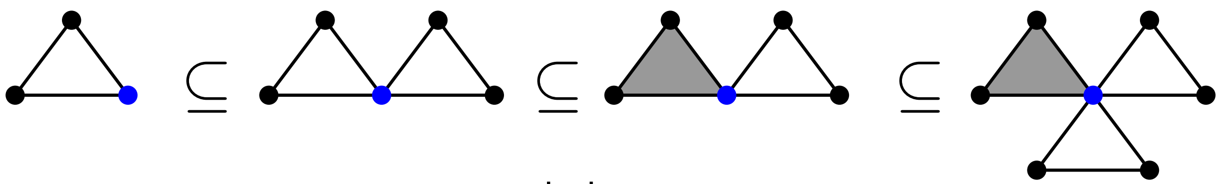
1-parameter persistence modules

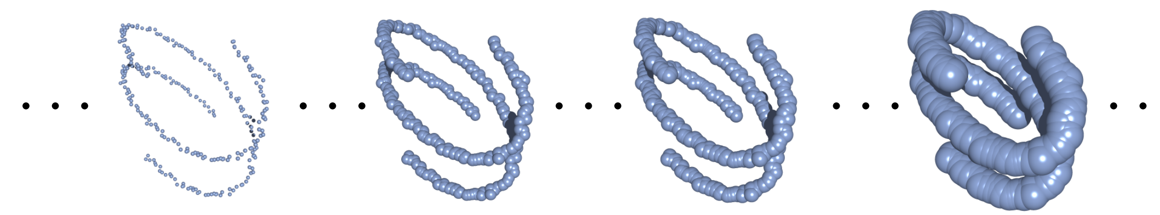
discrete setting: $M : [\![1,n]\!] \to \mathrm{vect}_{\boldsymbol{k}}$



$$\Downarrow H_1$$

$$\boldsymbol{k} \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} \boldsymbol{k}^2 \xrightarrow{[0\ 1]} \boldsymbol{k} \xrightarrow{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} \boldsymbol{k}^2$$

continuous setting: $M : \mathbb{R} \to \mathrm{vect}_{\boldsymbol{k}}$



$$\Downarrow H_0$$

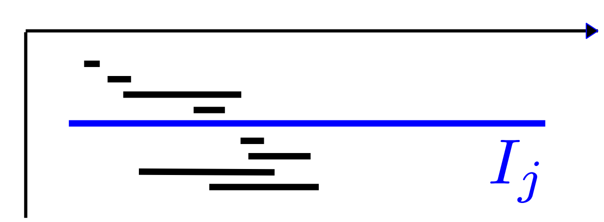
$$\cdots \to \boldsymbol{k}^{130} \to \cdots \to \boldsymbol{k}^2 \to \cdots \to \boldsymbol{k}^2 \to \cdots \to \boldsymbol{k} \to \cdots$$

Thm [Gabriel][Auslander][Ringel][Webb][Crawley-Boevey]

For any $P \subseteq \mathbb{R}$ and any $M : (P, \leq) \to \mathrm{vect}_{\boldsymbol{k}}$:

$$M \simeq \bigoplus_{j \in J} \boldsymbol{k}_{I_j}$$

where each $\mathrm{End}(\boldsymbol{k}_{I_j})$ is local

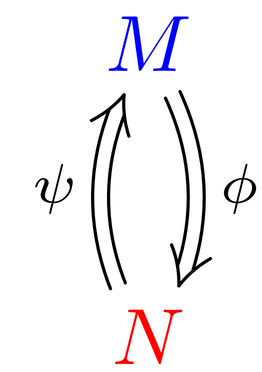


Bar M

Metric viewpoint: interleaving distance

Given $\color{blue}{M}, \color{red}{N} : (\mathbb{R}, \leq) \to \mathsf{vect}_k$,

- **morphism:** natural transformation $M \Rightarrow N$
- **isomorphism:** pair of morphisms $M \overset{\phi}{\Rightarrow} N$ and $N \overset{\psi}{\Rightarrow} M$ such that:



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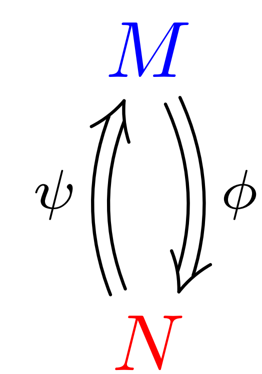
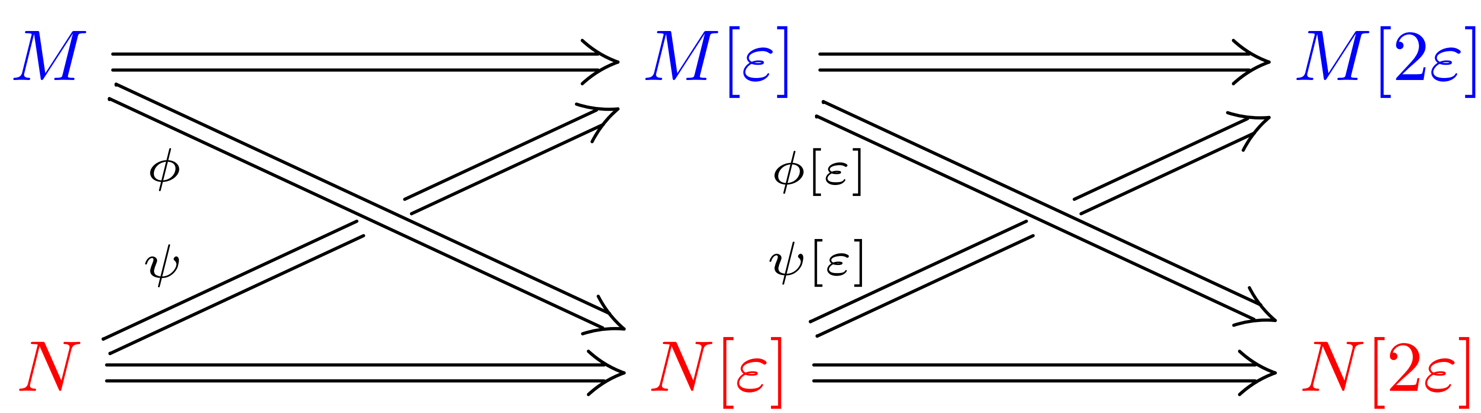
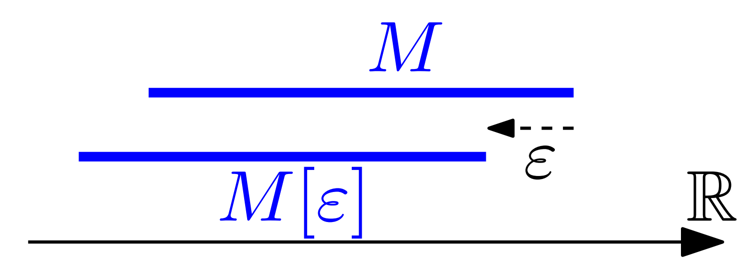


- **ε-isomorphism:** pair of morphisms $M \overset{\phi}{\Rightarrow} N[\varepsilon]$ and $N \overset{\psi}{\Rightarrow} M[\varepsilon]$ such that:





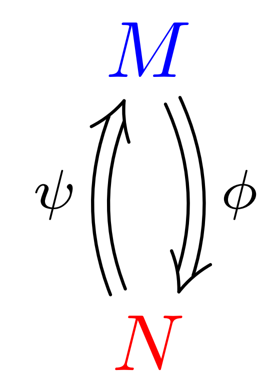
where $M[\varepsilon](t) := M(t+\varepsilon)$ and $\phi[\varepsilon](t) := \phi(t+\varepsilon)$

- **interleaving distance:** $\mathrm{d}_i(M,N) := \inf\{\varepsilon \mid M, N\ \varepsilon\text{-isomorphic}\}$

Metric viewpoint: interleaving distance

Given $M, N : (\mathbb{R}, \leq) \to \text{vect}_k$,

- **morphism:** natural transformation $M \Rightarrow N$
- **isomorphism:** pair of morphisms $M \overset{\phi}{\Rightarrow} N$ and $N \overset{\psi}{\Rightarrow} M$ such that:



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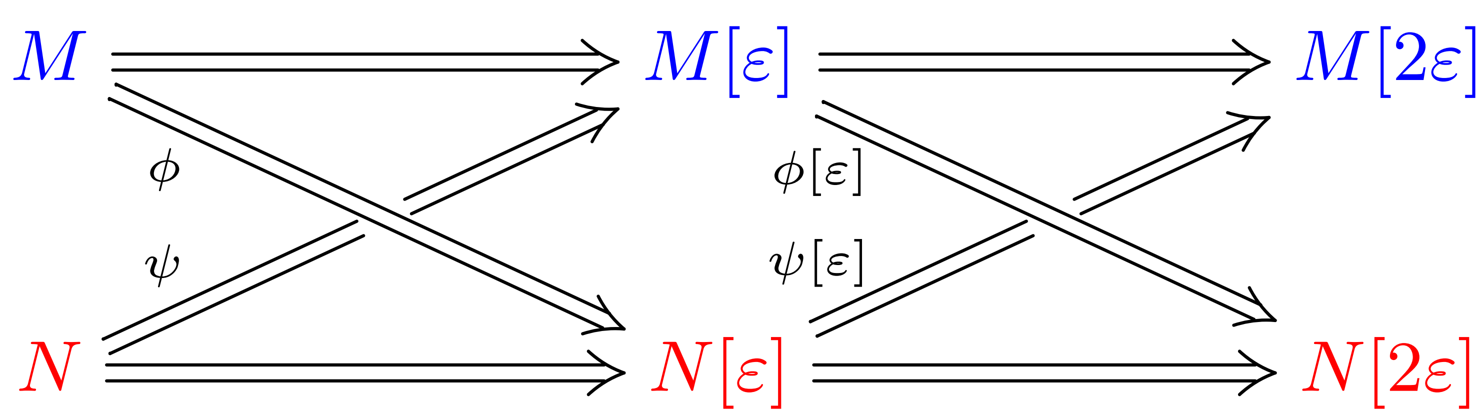


Prop: $\forall\ f, g : X \to \mathbb{R}$,
$\mathrm{d}_i(HF, HG) \leq \|f - g\|_\infty$

Thm: [Lesnick]
d stable as above $\Rightarrow \mathrm{d} \leq \mathrm{d}_i$

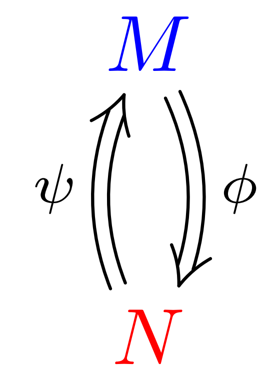
where $M[\varepsilon](t) := M(t + \varepsilon)$ and $\phi[\varepsilon](t) := \phi(t + \varepsilon)$

- **interleaving distance:** $\mathrm{d}_i(M, N) := \inf\{\varepsilon \mid M, N\ \varepsilon\text{-isomorphic}\}$

Metric viewpoint: bottleneck distance

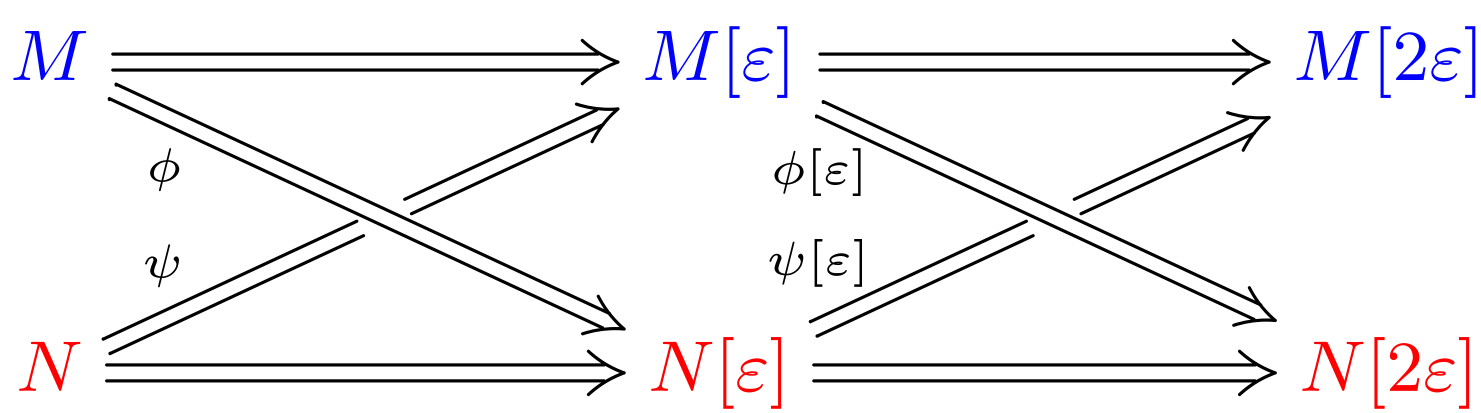
Given $M = \bigoplus_{a \in A} M_a$, $N = \bigoplus_{b \in B} N_b : (\mathbb{R}, \leq) \to \mathrm{vect}_k$,

- **morphism:** natural transformation $M \Rightarrow N$
- **isomorphism:** pair of morphisms $M \overset{\phi}{\Rightarrow} N$ and $N \overset{\psi}{\Rightarrow} M$ such that:

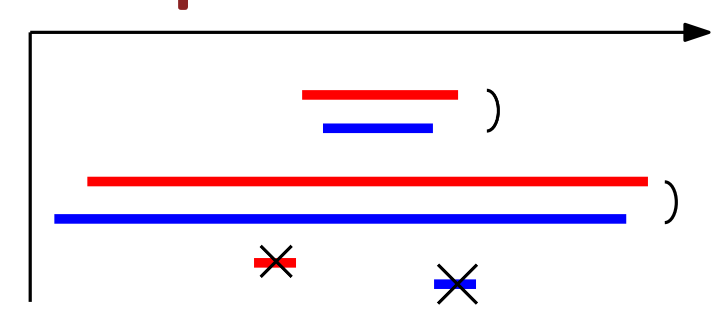


bottleneck

- **ε-isomorphism:** pair of morphisms $M \overset{\phi}{\Rightarrow} N[\varepsilon]$ and $N \overset{\psi}{\Rightarrow} M[\varepsilon]$ such that:



and ϕ, ψ factor through the decompositions of M and N



where $M[\varepsilon](t) := M(t+\varepsilon)$ and $\phi[\varepsilon](t) := \phi(t+\varepsilon)$

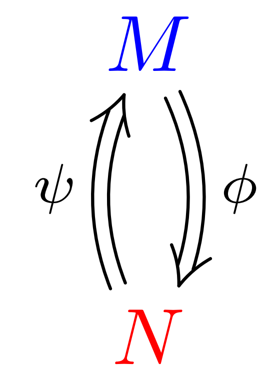
- **bottleneck distance:** $\mathrm{d_b}(M,N) := \inf\{\varepsilon \mid M, N \text{ bottleneck } \varepsilon\text{-isomorphic}\}$

Metric viewpoint: isometry

Given $M = \bigoplus_{a \in A} M_a$, $N = \bigoplus_{b \in B} N_b : (\mathbb{R}, \leq) \to \mathsf{vect}_k$,

- **morphism:** natural transformation $M \Rightarrow N$

- **isomorphism:** pair of morphisms $M \overset{\phi}{\Rightarrow} N$ and $N \overset{\psi}{\Rightarrow} M$ such that:



bottleneck

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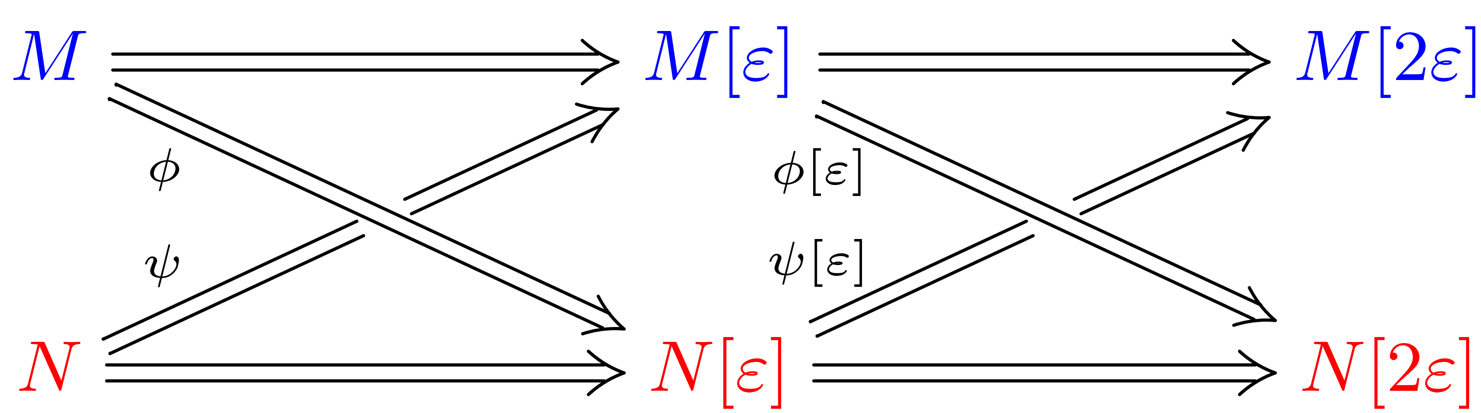


and ϕ, ψ factor through the decompositions of M and N

Thm: $\mathrm{d}_i = \mathrm{d}_\mathrm{b}$

where $M[\varepsilon](t) := M(t+\varepsilon)$ and $\phi[\varepsilon](t) := \phi(t+\varepsilon)$

bottleneck

- **bottleneck distance:** $\mathrm{d}_\mathrm{b}(M,N) := \inf\{\varepsilon \mid M,N\ \varepsilon\text{-isomorphic}\}$

Wrap-up on 1-parameter persistence modules

Structure theorems

- complete classification of pfd persistence modules via their barcodes
- efficient algorithms for barcode computation

Isometry theorem

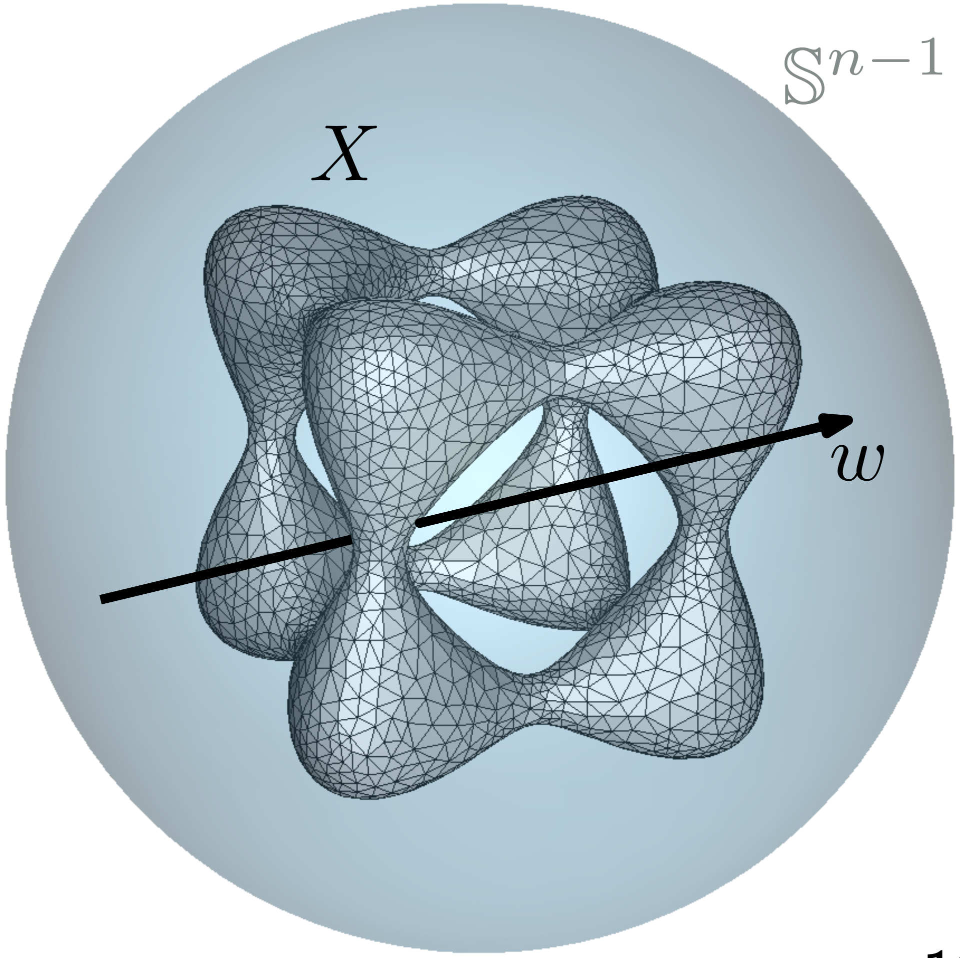
- barcodes as complete metric invariants for persistence modules
  - $\to$ combinatorial algorithms for distance computation
- space of barcodes as a space of measures
  - $\to$ bounds on intrinsic curvature
  - $\to$ toolbox for statistics
  - $\to$ vectorizations and kernels for ML

Multi-parameter persistence... what for?

- study joint variables (e.g. treatment efficacy vs. risk)

- increase feature sensitivity (by enhanced feature aggregation)

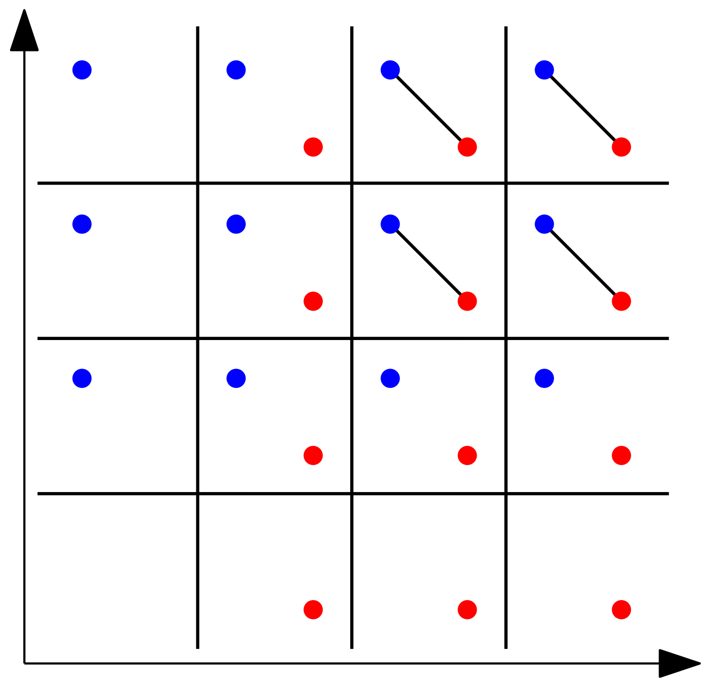
Thm: [Boyer, Curry, Mukherjee, Turner]
[Ghrist, Levanger, Mai]
The map $X \mapsto \{\mathrm{Bar}\,\langle \cdot, w\rangle|_X\}_{w\in\mathbb{S}^{n-1}}$ is injective on the class of compact subanalytic sets $X \subset \mathbb{R}^n$.



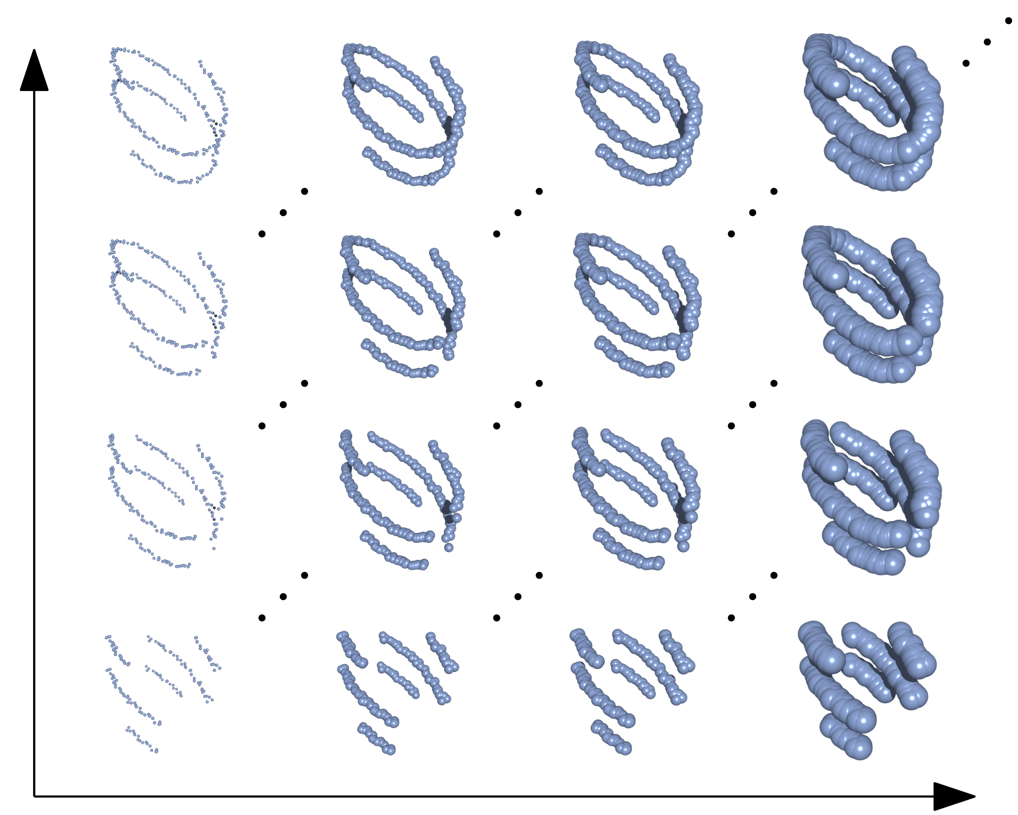
Q: can we reduce to finitely many directions using multi-parameter persistence?

Multi-parameter persistence modules

discrete setting: $M : [\![1,n]\!]^d \to \mathsf{vect}_{\boldsymbol{k}}$



continuous setting: $M : \mathbb{R}^d \to \mathsf{vect}_{\boldsymbol{k}}$



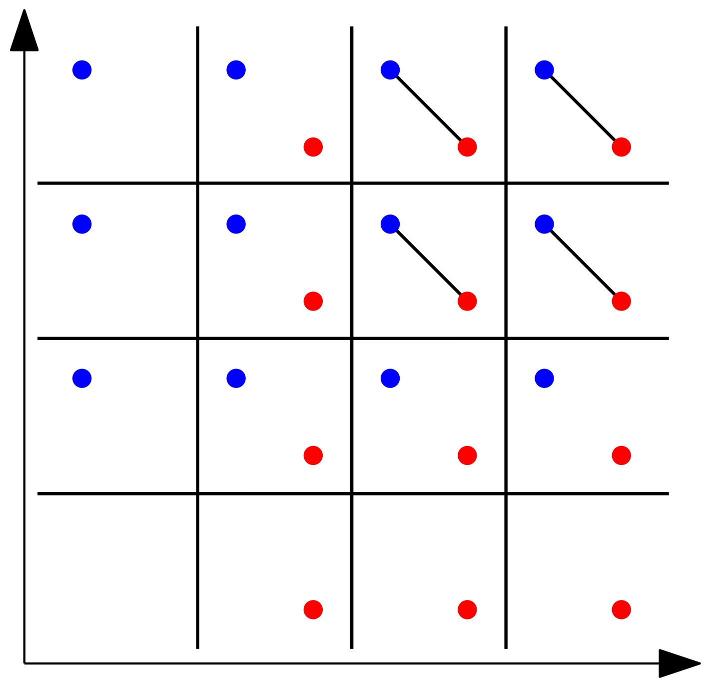
Thm [Botnan, Crawley-Boevey]

For any poset $(P, \leq)$ and functor $M : (P, \leq) \to \mathsf{vect}_{\boldsymbol{k}}$:

$$M \simeq \bigoplus_{j \in J} M_j \quad \text{where each } \mathrm{End}(M_j) \text{ is local}$$

Multi-parameter persistence modules

discrete setting: $M : [\![1,n]\!]^d \to \mathrm{vect}_{\boldsymbol{k}}$



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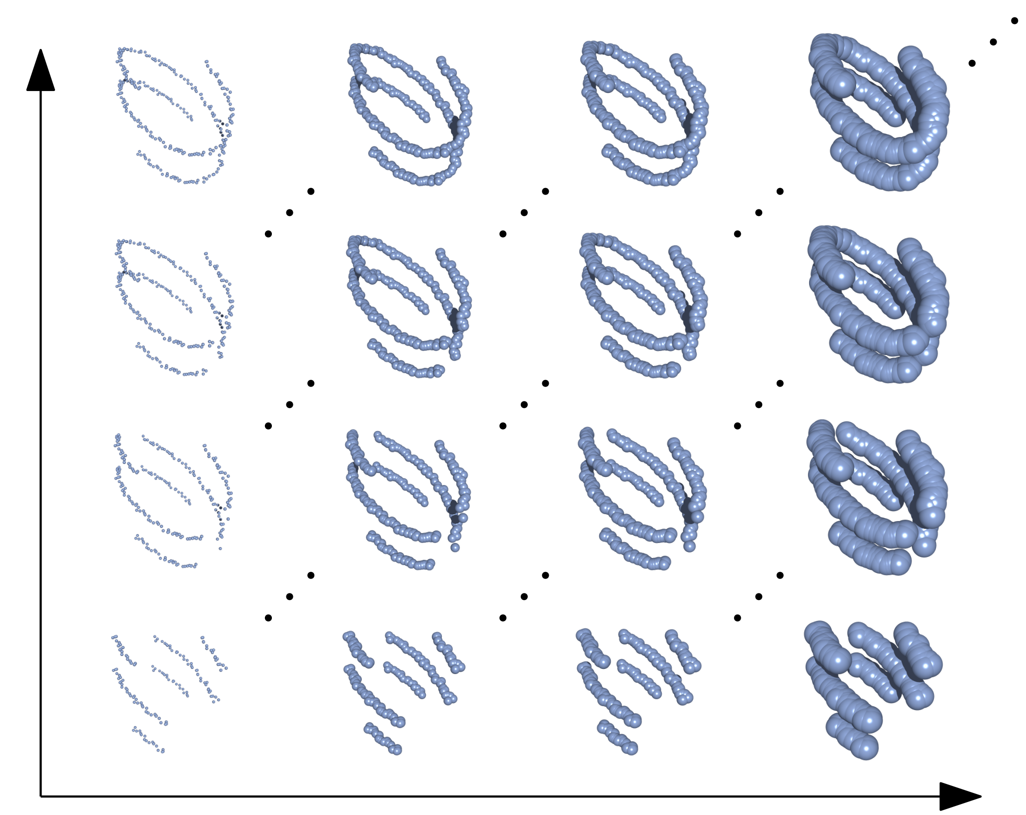
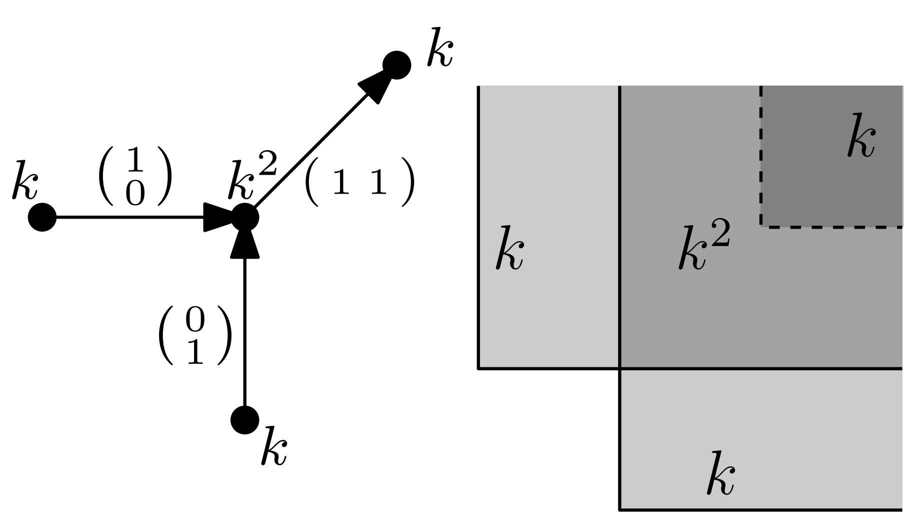
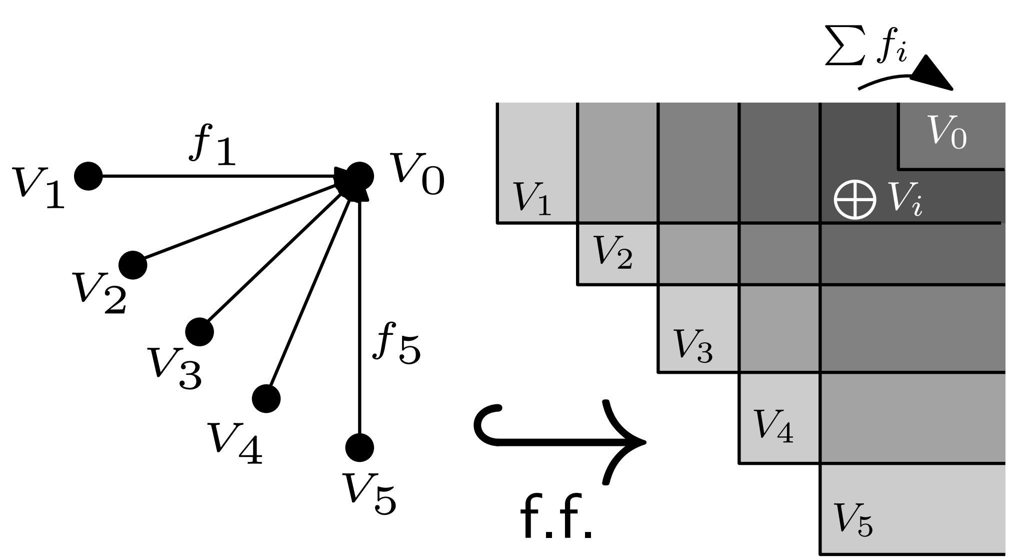




non-thin summands



wild-type

Multi-parameter persistence modules

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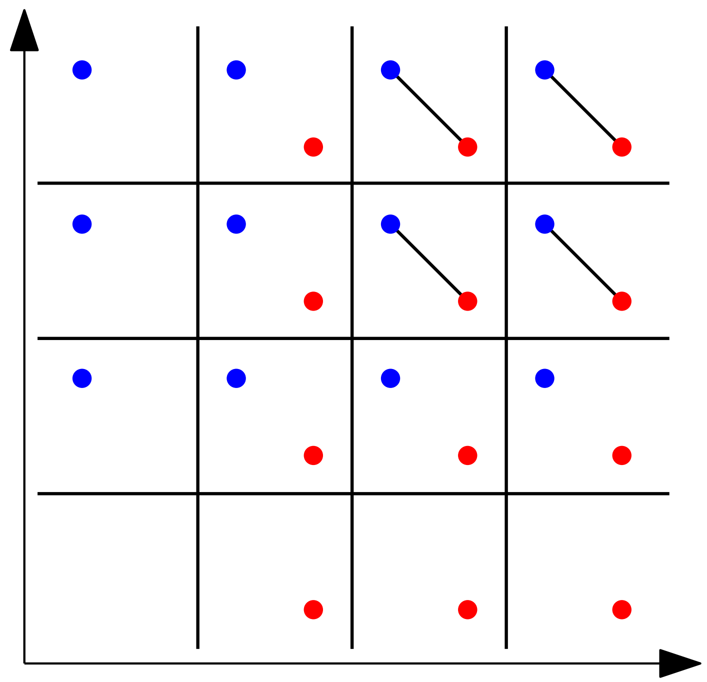

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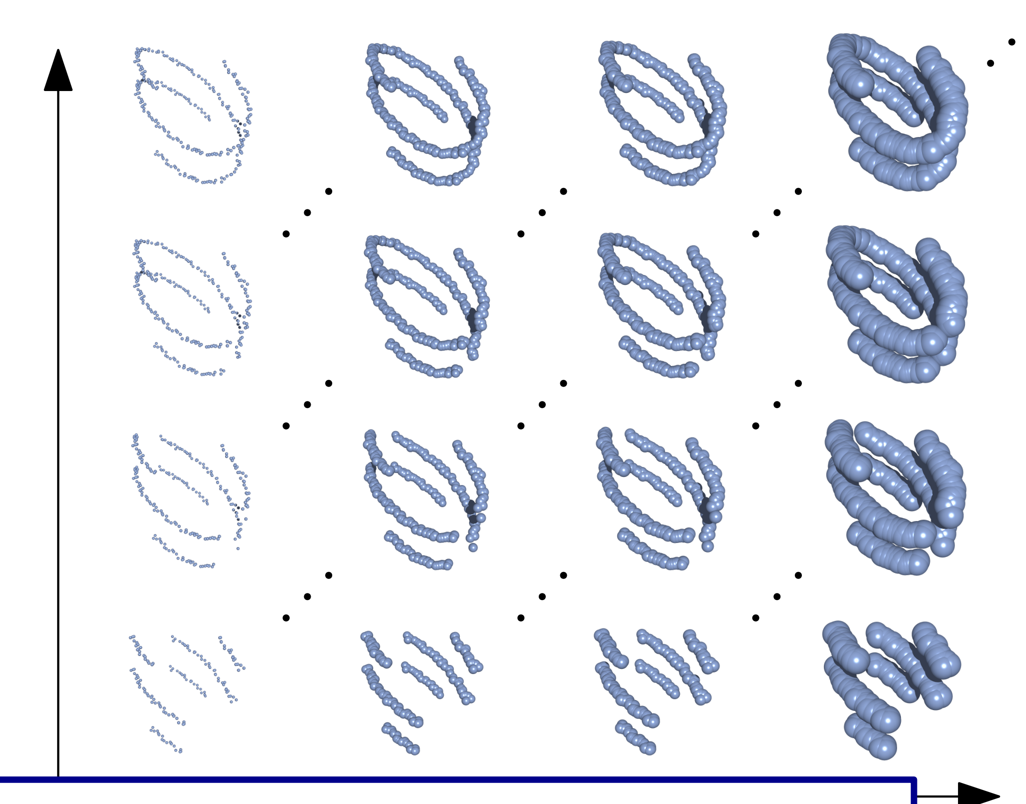
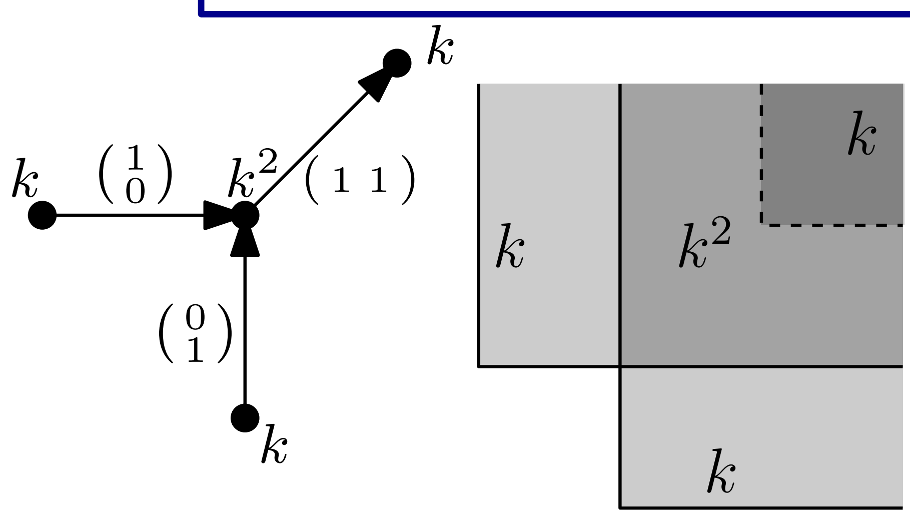
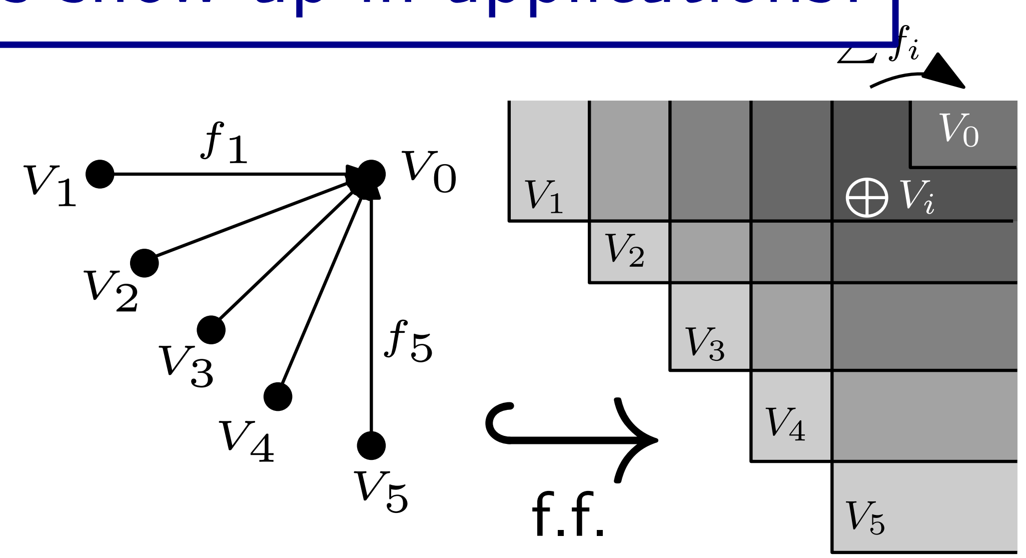


Q: do non-thin indecomposables show up in applications?

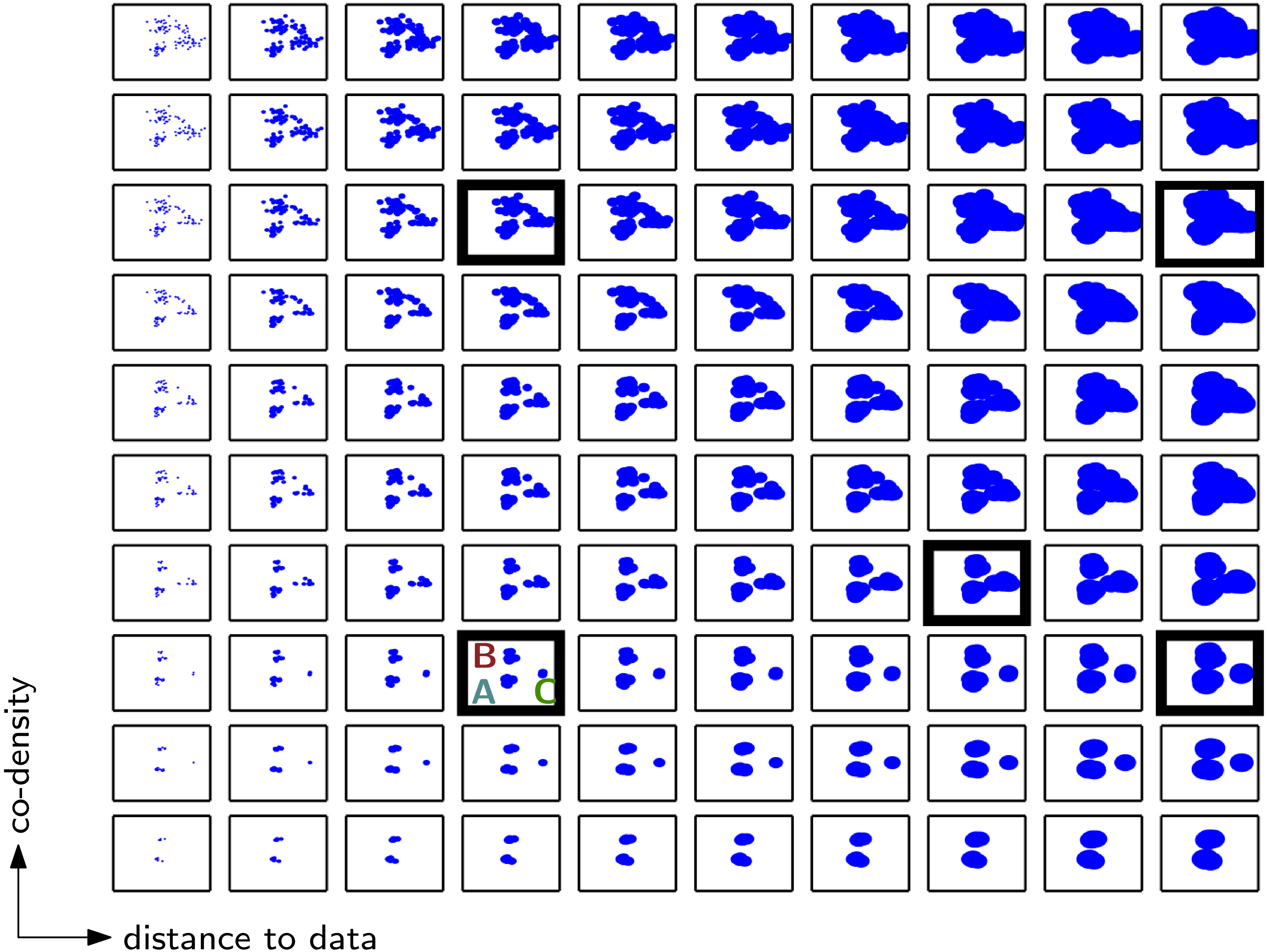


non-thin summands



wild-type

Example: two-parameter clustering



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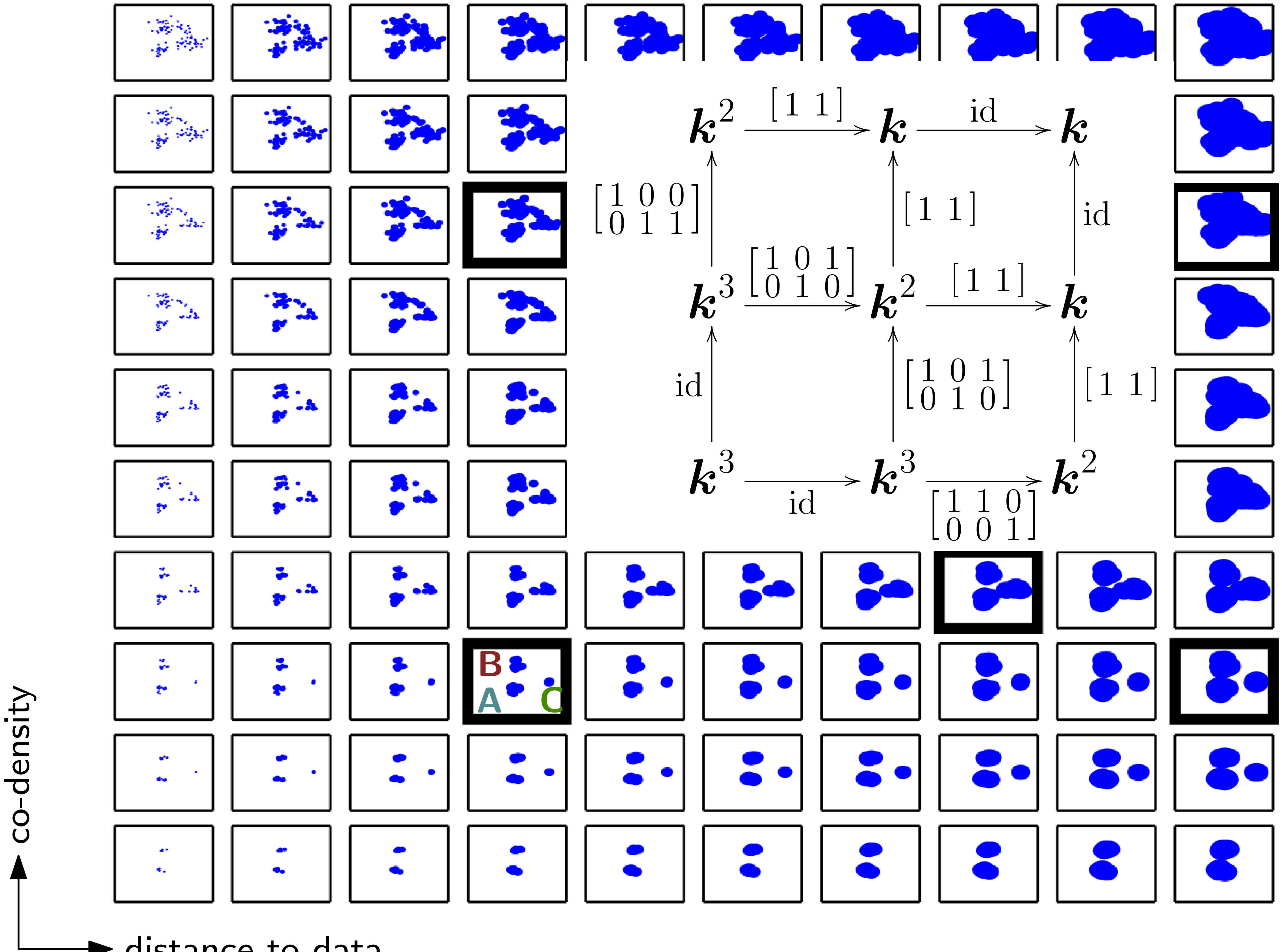

Example: two-parameter clustering

$$
\begin{array}{ccccccccc}
M(1,m) & \twoheadrightarrow & M(2,m) & \twoheadrightarrow & \cdots & \twoheadrightarrow & M(n-1,m) & \twoheadrightarrow & M(n,m) \\
\uparrow & & \uparrow & & & & \uparrow & & \uparrow \\
M(1,m-1) & \twoheadrightarrow & M(2,m-1) & \twoheadrightarrow & \cdots & \twoheadrightarrow & M(n-1,m-1) & \twoheadrightarrow & M(n,m-1) \\
\uparrow & & \uparrow & & & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \text{(after applying } H_0\text{)} & & \vdots & & \vdots \\
\uparrow & & \uparrow & & & & \uparrow & & \uparrow \\
M(1,2) & \twoheadrightarrow & M(2,2) & \twoheadrightarrow & \cdots & \twoheadrightarrow & M(n-1,2) & \twoheadrightarrow & M(n,2) \\
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\end{array}
$$

(after applying H_0)

Thm: [Bauer, Botnan, Oppermann, Steen]

$$\frac{\mathrm{Fun}^{e,*}([\![1,n]\!] \times [\![1,m]\!], \mathsf{vect}_{\boldsymbol{k}})}{\mathrm{Fun}^{e,m}([\![1,n]\!] \times [\![1,m]\!], \mathsf{vect}_{\boldsymbol{k}})} \simeq \mathrm{Fun}([\![1,n]\!] \times [\![1,m-1]\!], \mathsf{vect}_{\boldsymbol{k}})$$

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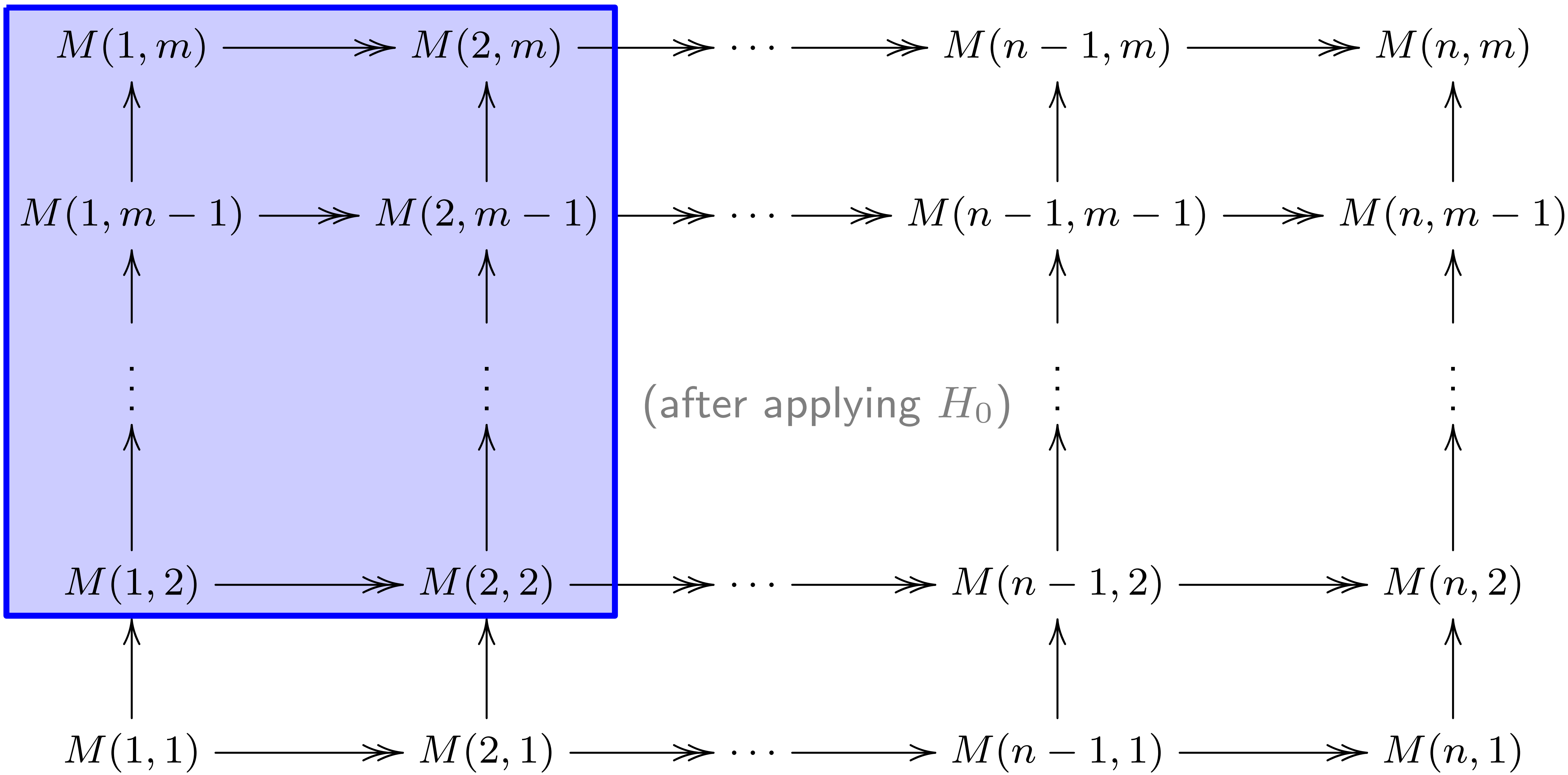

Thm: [Bauer, Botnan, Oppermann, Steen]

modules decompose into summands $\boldsymbol{k}_{[\![1,i]\!]\times[\![j,m]\!]}$

$$\frac{\mathrm{Fun}^{e,*}([\![1,n]\!]\times[\![1,m]\!],\mathsf{vect}_{\boldsymbol{k}})}{\mathrm{Fun}^{e,m}([\![1,n]\!]\times[\![1,m]\!],\mathsf{vect}_{\boldsymbol{k}})} \simeq \mathrm{Fun}([\![1,n]\!]\times[\![1,m-1]\!],\mathsf{vect}_{\boldsymbol{k}})$$

Incomplete invariants

Bottomline: look for incomplete invariants of persistence modules that are:

- as strong as possible (μ stronger than ν if $\mu(M) = \mu(N) \Rightarrow \nu(M) = \nu(N)$)
- manageable to compute (polynomial time in the input filtration size)
- stable w.r.t. perturbations of the modules in the interleaving distance d_i
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Examples:

- dimension vector / Hilbert function
- rank invariant
- global rank function / generalized rank invariant
- graded Betti numbers
- $\ldots$

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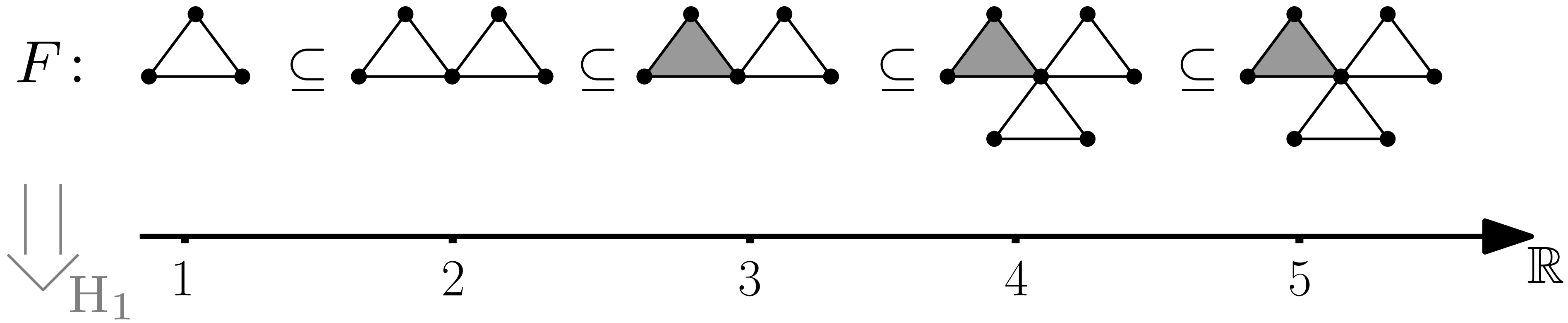
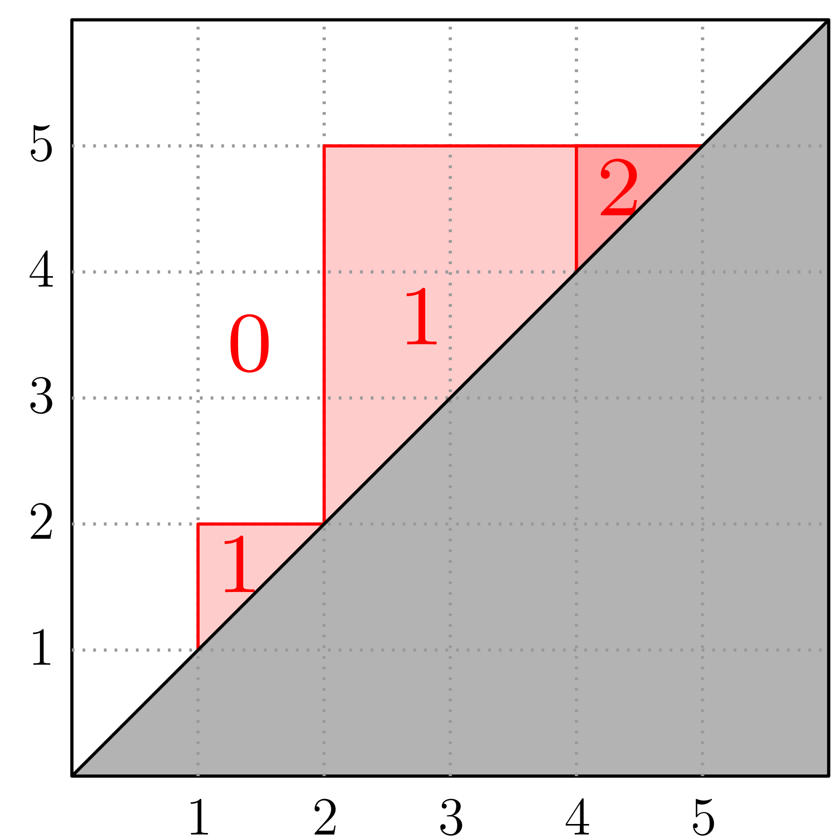
The rank invariant of 1-parameter modules

$P = [\![1, 5]\!] \subseteq (\mathbb{R}, \leq)$

$F:$



$$M: \quad \boldsymbol{k} \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} \boldsymbol{k}^2 \xrightarrow{\begin{pmatrix}0 & 1\end{pmatrix}} \boldsymbol{k} \xrightarrow{\begin{pmatrix}0\\1\end{pmatrix}} \boldsymbol{k}^2 \xrightarrow{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}} \boldsymbol{k}^2$$



$\mathrm{Rk}\, M : (s \leq t) \mapsto \mathrm{rank}\, [M_s \to M_t]$

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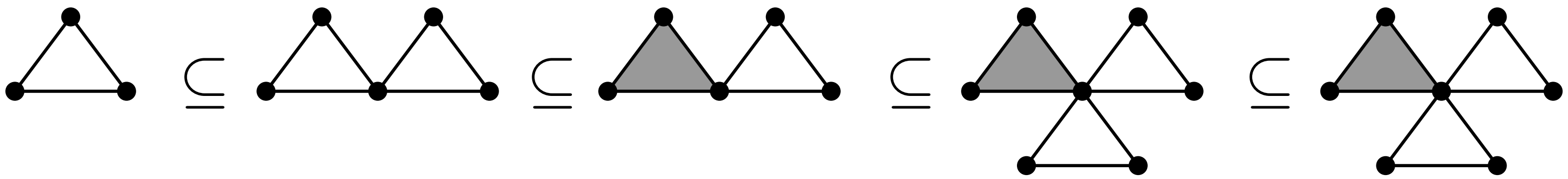

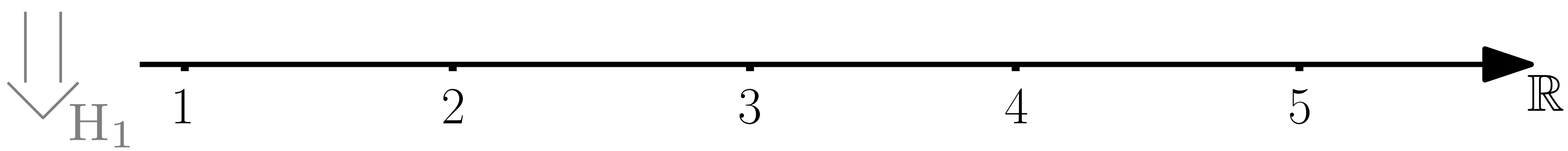

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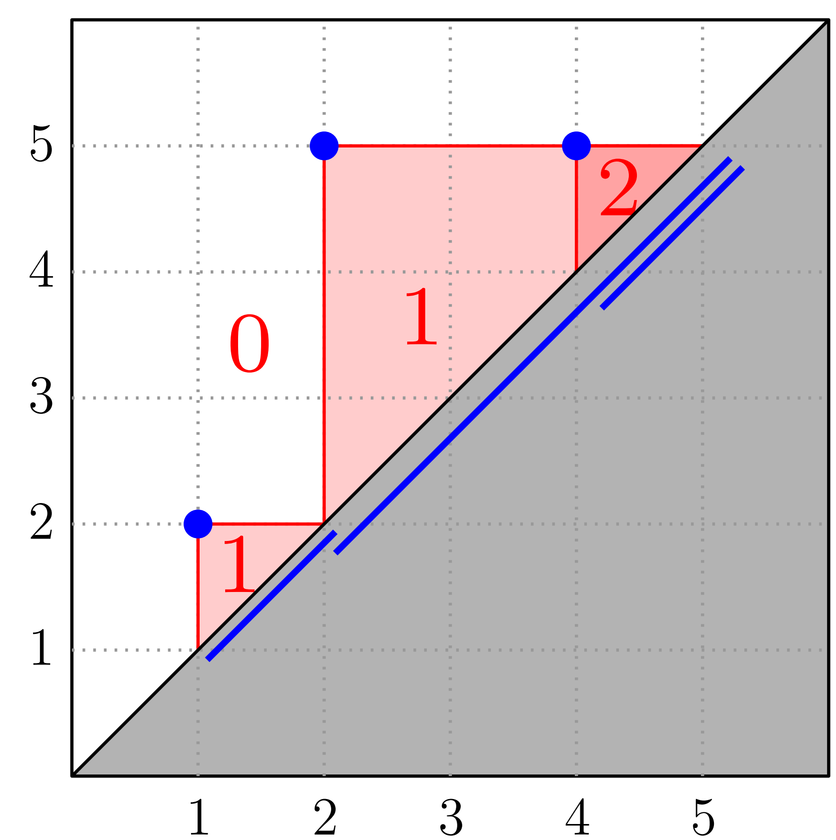

$$\mathrm{Rk}\, M = \sum_{I \in \mathrm{Bar}\, M} \mathrm{Rk}\, \boldsymbol{k}_I = \mathrm{Rk} \left(\bigoplus_{I \in \mathrm{Bar}\, M} \boldsymbol{k}_I \right)$$

(rank invariant $\mathbb{N}$-decomposes on interval rank funcions)

The rank invariant of 1-parameter modules

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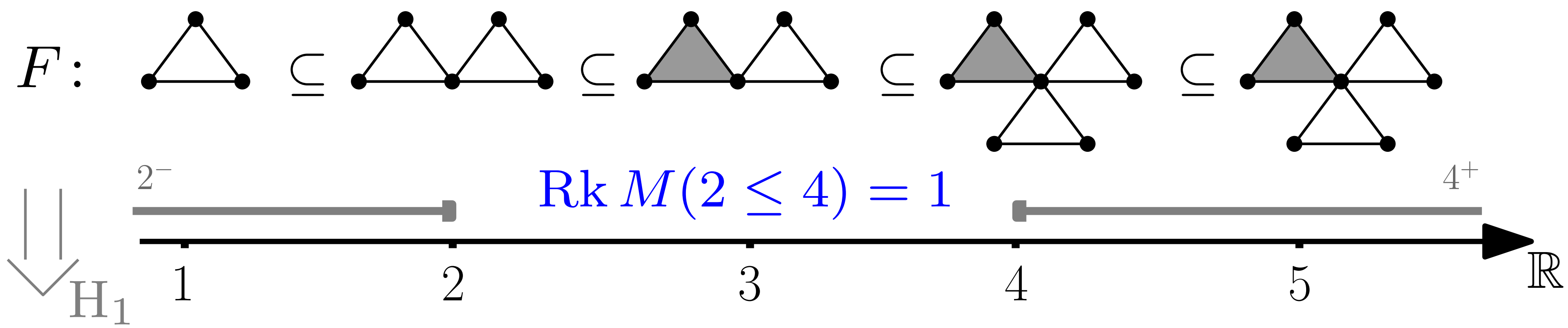
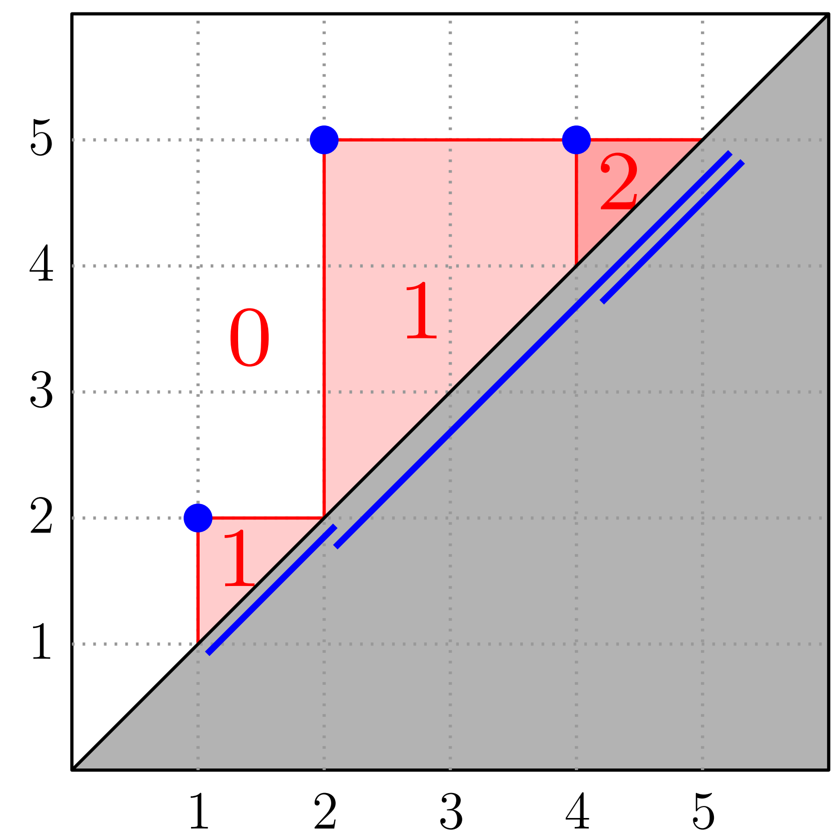


$$M: \quad \boldsymbol{k} \xrightarrow{\binom{1}{0}} \boldsymbol{k}^2 \xrightarrow{(0\ 1)} \boldsymbol{k} \xrightarrow{\binom{0}{1}} \boldsymbol{k}^2 \xrightarrow{\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)} \boldsymbol{k}^2$$



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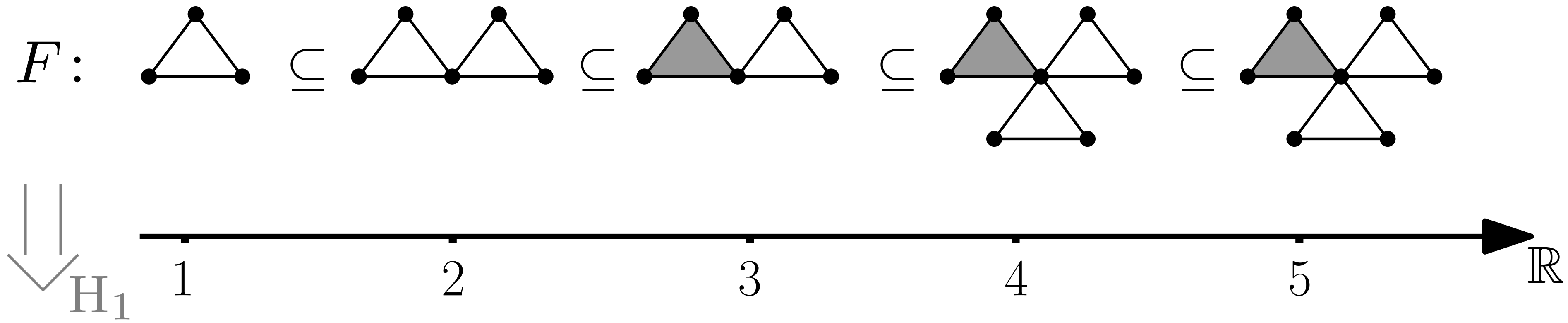

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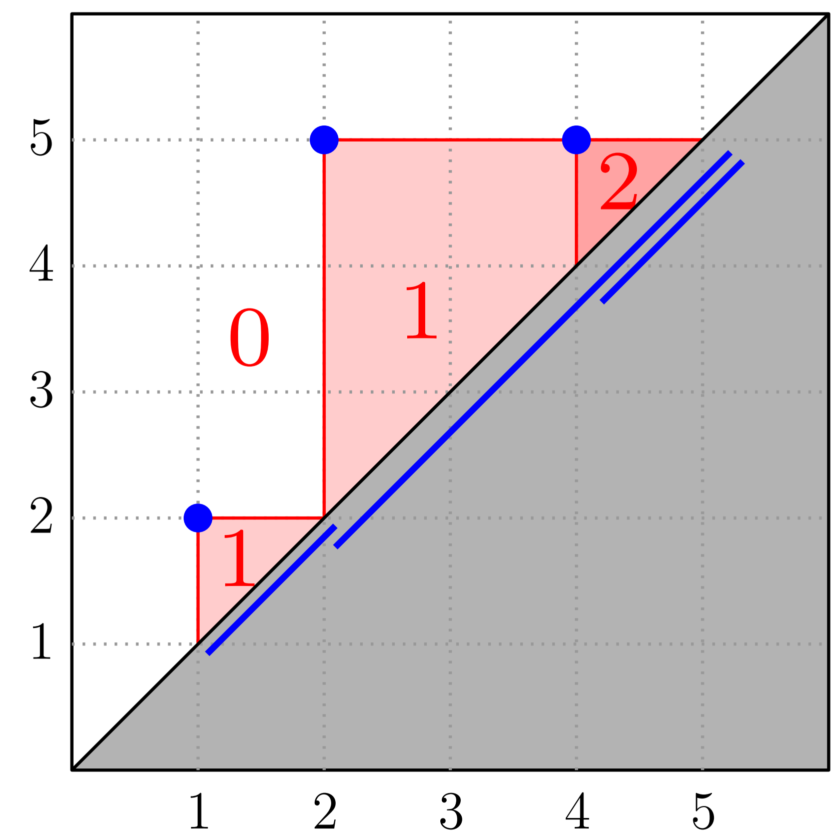
$\mathrm{Bar}\, M:$



$$\mathrm{Rk}\, M = \sum_{I \in \mathrm{Bar}\, M} \mathrm{Rk}\, \boldsymbol{k}_I = \mathrm{Rk} \left(\bigoplus_{I \in \mathrm{Bar}\, M} \boldsymbol{k}_I \right)$$

(unique decomposition, terms match with summands of M)

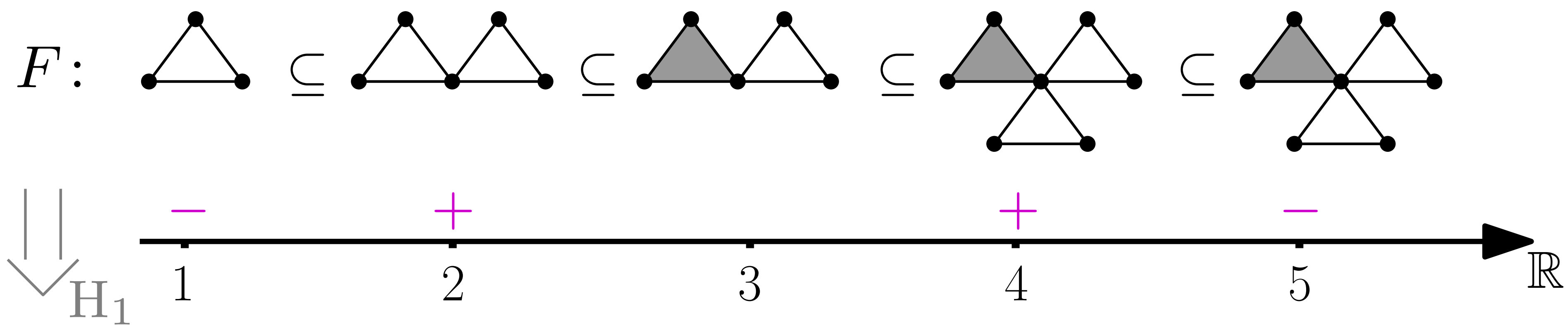
$$M \simeq \bigoplus_{I \in \mathrm{Bar}\, M} \boldsymbol{k}_I$$

($\Rightarrow$ rank invariant is complete)

The rank invariant of 1-parameter modules

$P = [\![1,5]\!] \subseteq (\mathbb{R}, \leq)$

$F:$



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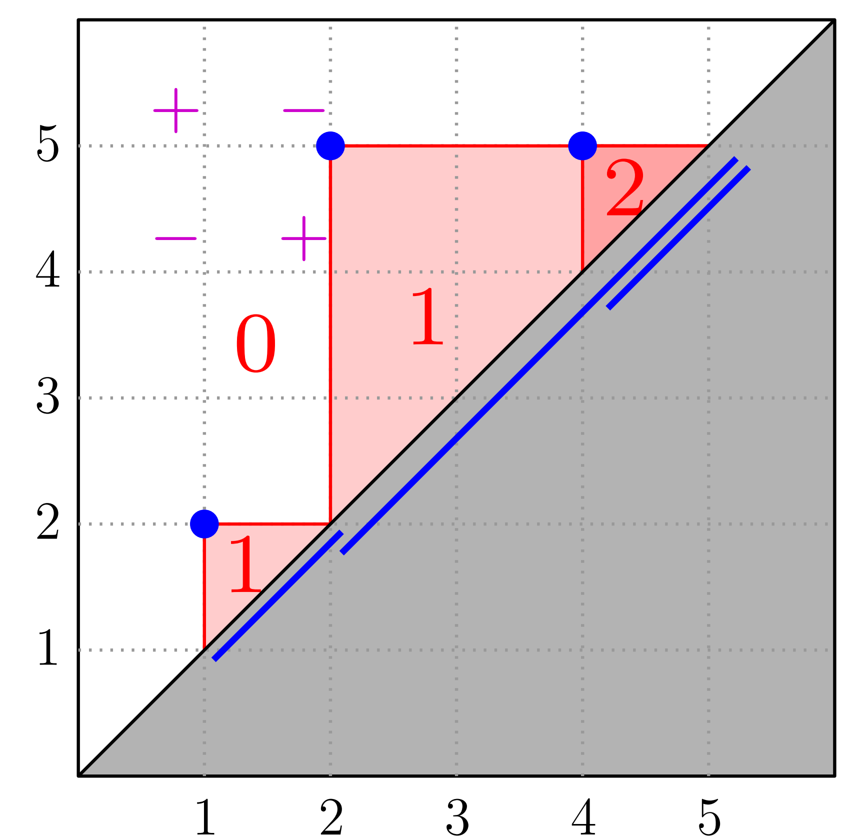


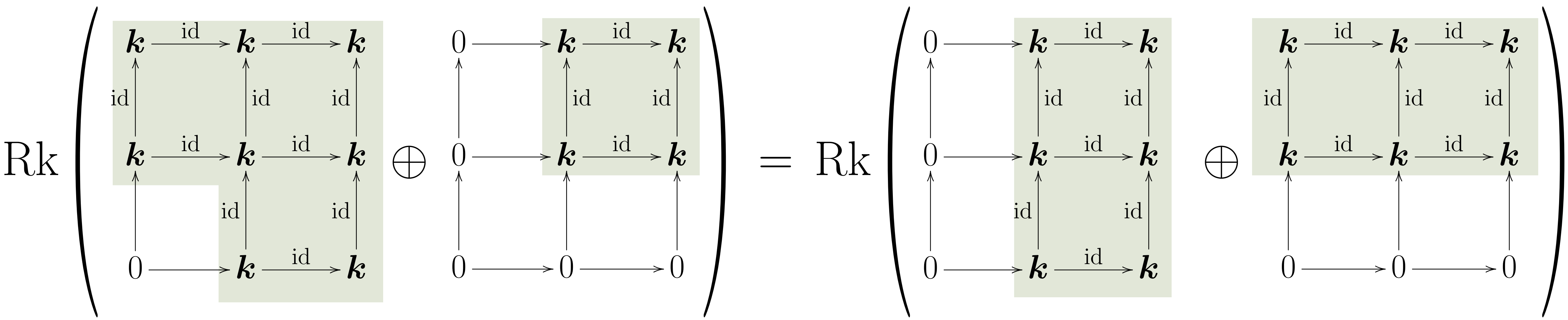
$$\color{red}{\mathrm{Rk}\, M} = \sum_{I \in \mathrm{Bar}\, M} \mathrm{Rk}\, \boldsymbol{k}_I = \mathrm{Rk}\left(\bigoplus_{I \in \mathrm{Bar}\, M} \boldsymbol{k}_I \right)$$

$$\mathrm{mult}_{[\![i,j]\!]}\, \mathrm{Bar}\, M = \mathrm{Rk}\, M(i,j) - \mathrm{Rk}\, M(i-1,j) - \mathrm{Rk}\, M(i,j+1) + \mathrm{Rk}\, M(i-1,j+1)$$

The rank invariant of multi-parameter modules

$P = [\![1,3]\!]^2 \subseteq (\mathbb{R}^2, \leq)$

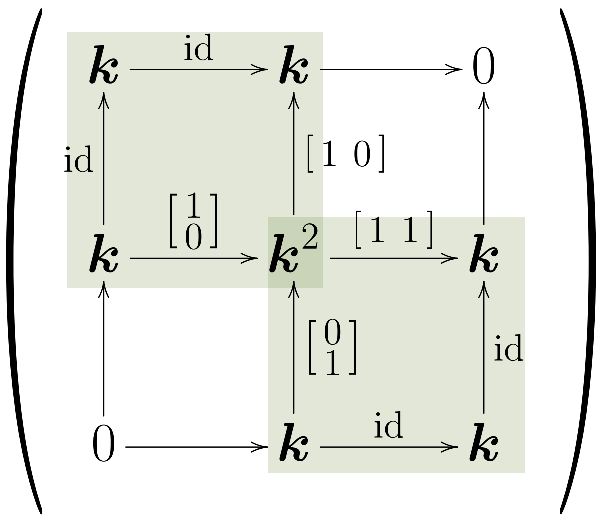
$\mathrm{Rk}\, M : (s \leq t) \mapsto \mathrm{rank}\,[M_s \to M_t]$



(rank invariant is not complete)

The rank invariant of multi-parameter modules

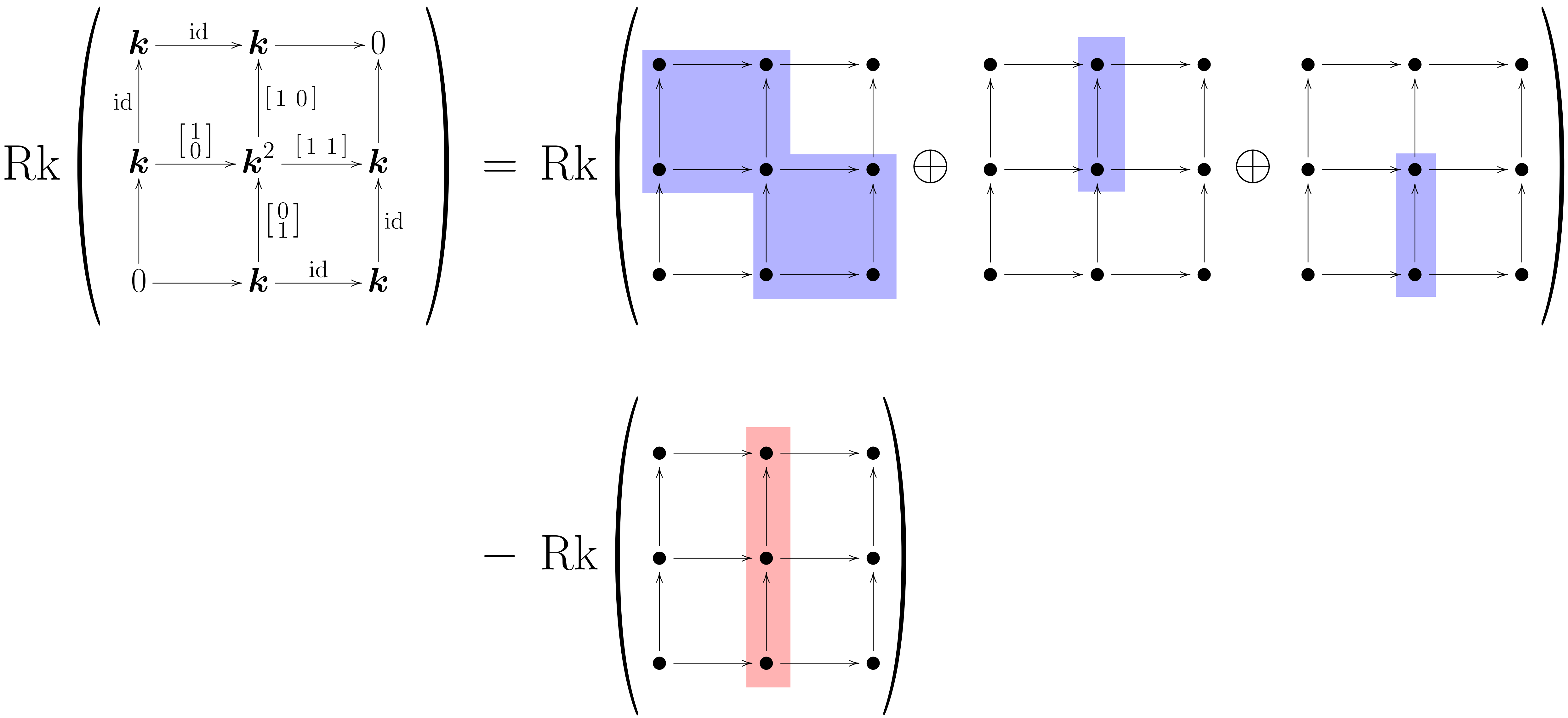
$P = [\![1,3]\!]^2 \subseteq (\mathbb{R}^2, \leq)$

Rk  $= \mathrm{Rk}\left(\quad ? \quad \right)$

(rank invariant does not $\mathbb{N}$-decompose on interval rank functions)

The rank invariant of multi-parameter modules

$P = [\![1,3]\!]^2 \subseteq (\mathbb{R}^2, \leq)$



(rank invariant $\mathbb{Z}$-decomposes on interval rank functions)

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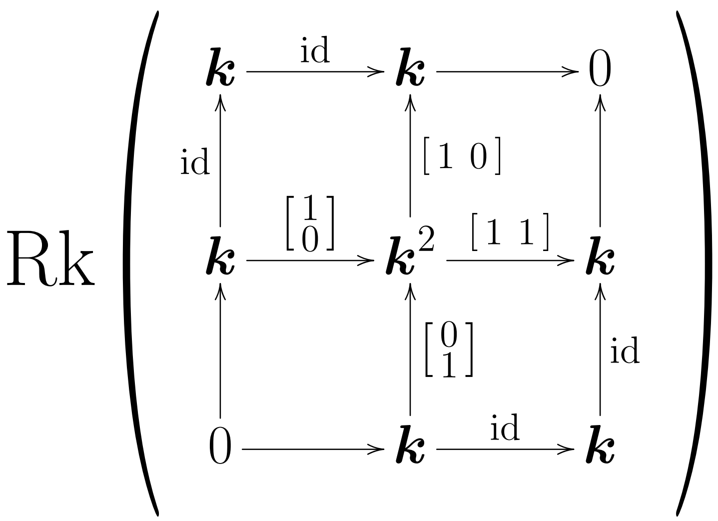

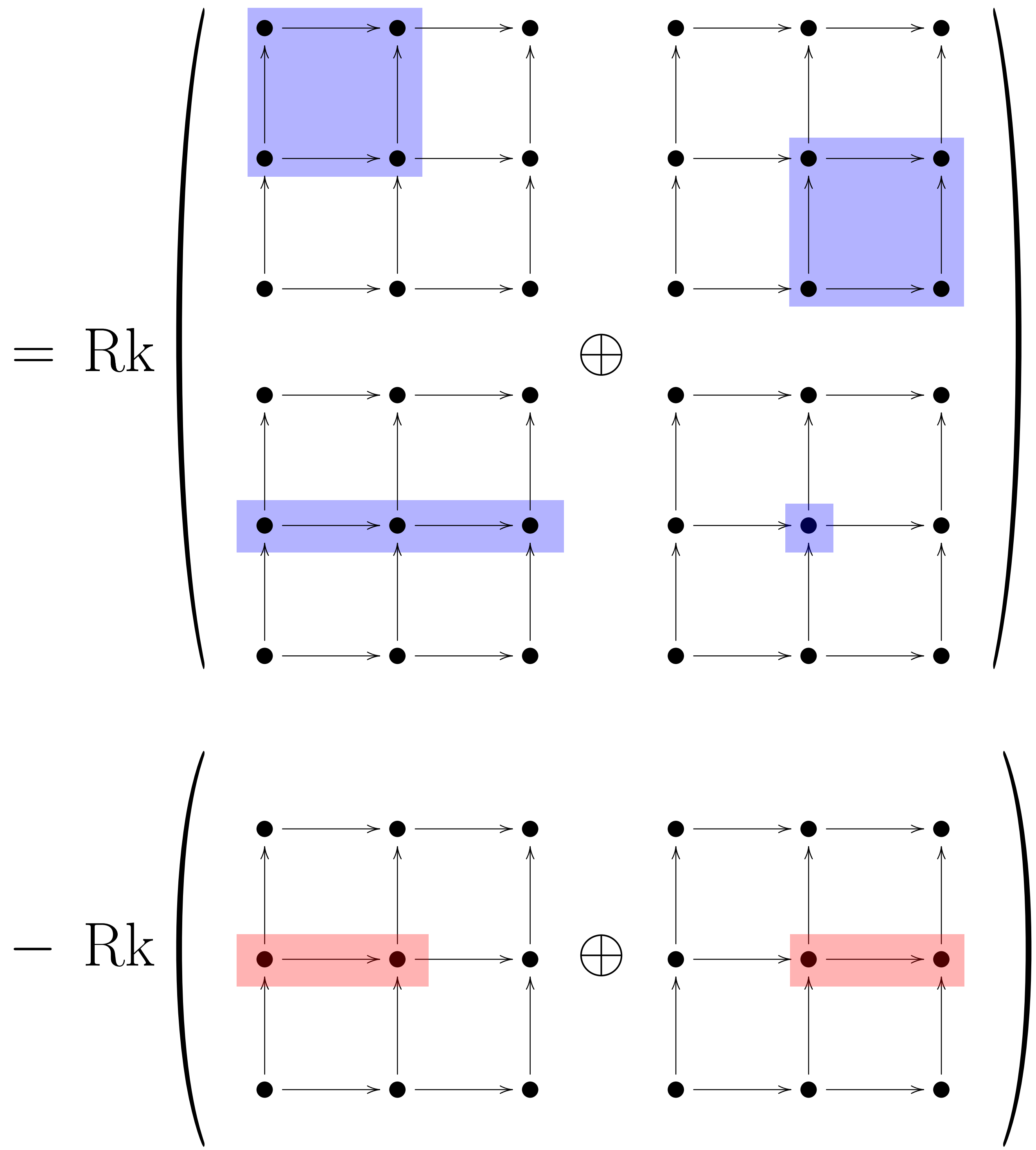

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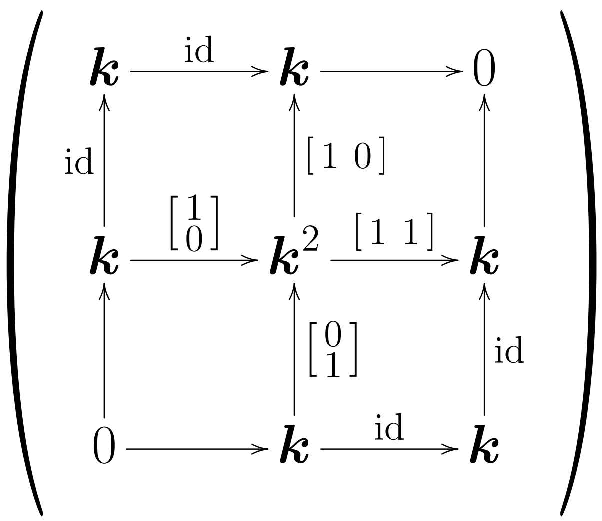
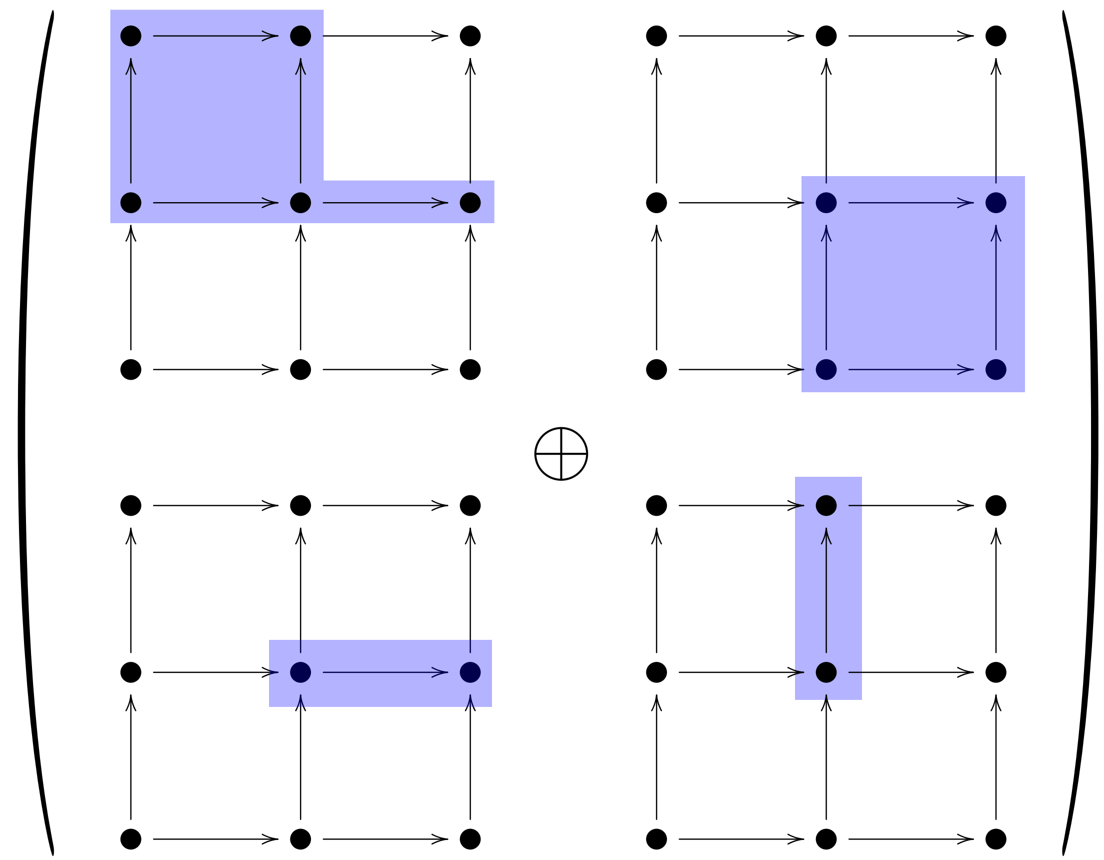
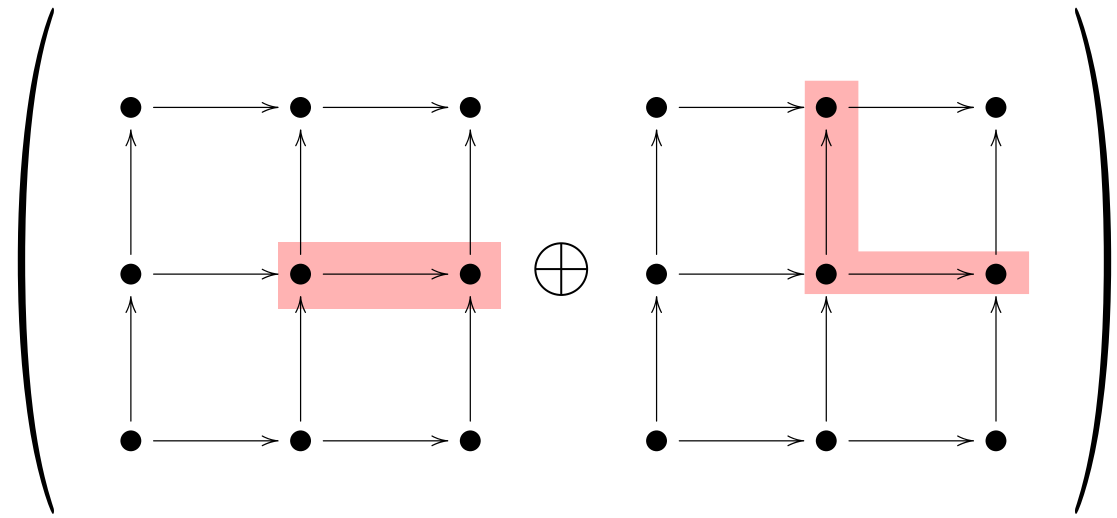
Rk 

$= \mathrm{Rk}$ 

$- \mathrm{Rk}$ 

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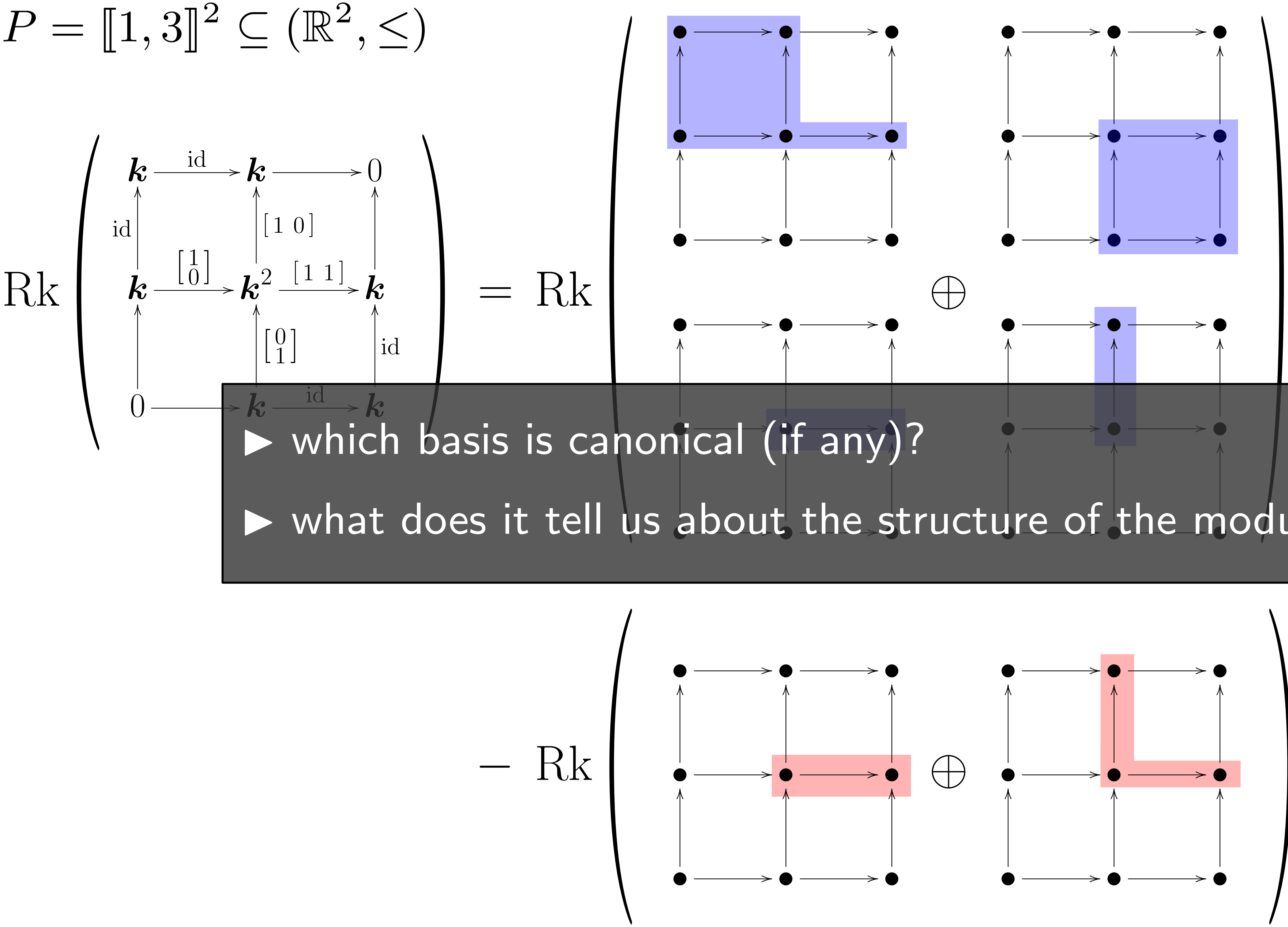


- which basis is canonical (if any)?
- what does it tell us about the structure of the module?

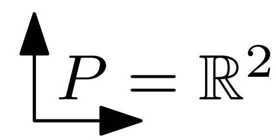
(rank invariant $\mathbb{Z}$-decomposes on interval rank functions)

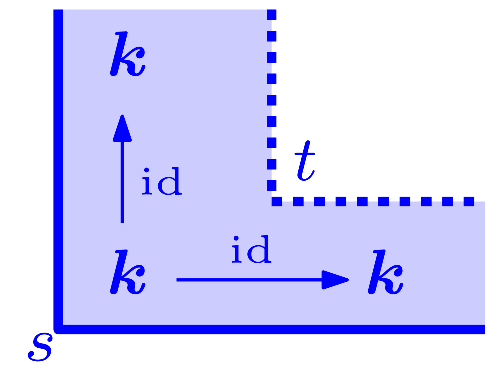
Canonical basis for the rank invariant

Hook / **Upset**: for $s < t \in P \cup \{\infty\}$,

$$\langle s, t \langle = \{u \in P \mid s \leq u \not\geq t\}$$

$$s^+ = \langle s, \infty \langle$$





Canonical basis for the rank invariant

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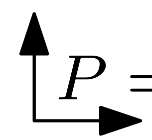
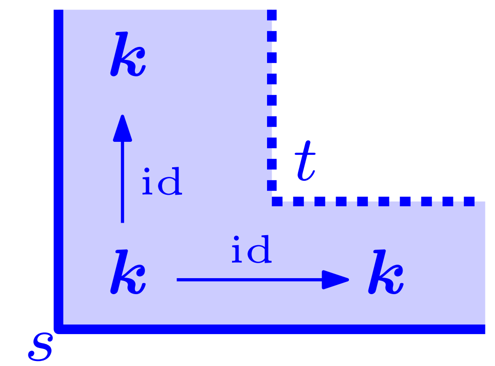


$P = \mathbb{R}^2$



Theorem ([Botnan, Oppermann, O.])**:**
If P is finite or an upper semi-lattice, then

$$\{\mathrm{Rk}\, \boldsymbol{k}_{\langle i,j\langle} \mid i < j \in P \cup \{\infty\}\}$$

generates uniquely

$$\{\mathrm{Rk}\, M \mid M \colon P \to \text{vect}_{\boldsymbol{k}} \text{ finitely presentable (fp)}\}$$

via projective resolutions relative to the rank-exact structure $\mathcal{E}_{\mathrm{Rk}}$.

The rank-exact structure

Let $\mathcal{E}_{\mathrm{Rk}}$ be the collection of short *rank-exact* sequences of pfd persistence modules:

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \quad \text{s.t.} \quad \mathrm{Rk}\, B = \mathrm{Rk}\, A + \mathrm{Rk}\, C$$

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Proposition: (follows in part from [Auslander, Solberg])

- $\mathcal{E}_{\mathrm{Rk}}$ defines the structure of an exact category on $\mathrm{vect}_{\boldsymbol{k}}^{P}$
- the indecomposable projectives relative to $\mathcal{E}_{\mathrm{Rk}}$ are the hook modules
- P finite $\implies$ | $\mathcal{E}_{\mathrm{Rk}}$ has enough projectives | every $M \in \mathrm{vect}_{\boldsymbol{k}}^{P}$ has a finite projective resolution
- P upper semi-lattice $\implies$ | $\mathcal{E}_{\mathrm{Rk}}^{\mathrm{fp}}$ has enough projectives | every fp $M \in \mathrm{Vect}_{\boldsymbol{k}}^{P}$ has a finite projective resolution

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$$[B] = [A] + [C]$$

Corollary:

- P finite $\Longrightarrow$ the $\left[\boldsymbol{k}_{\langle i,j\langle}\right]$ for $i < j \in P \cup \{\infty\}$ generate $K_0(\mathcal{E}_{\mathrm{Rk}})$
- P upper semi-lattice $\Longrightarrow$ the $\left[\boldsymbol{k}_{\langle i,j\langle}\right]$ for $i < j \in P \cup \{\infty\}$ generate $K_0(\mathcal{E}_{\mathrm{Rk}}^{\mathrm{fp}})$

Given a finite rank-exact projective resolution $M_\bullet$ of M:

$$[M] = \sum_{i=0}^{\infty} (-1)^i \, [M_i]$$

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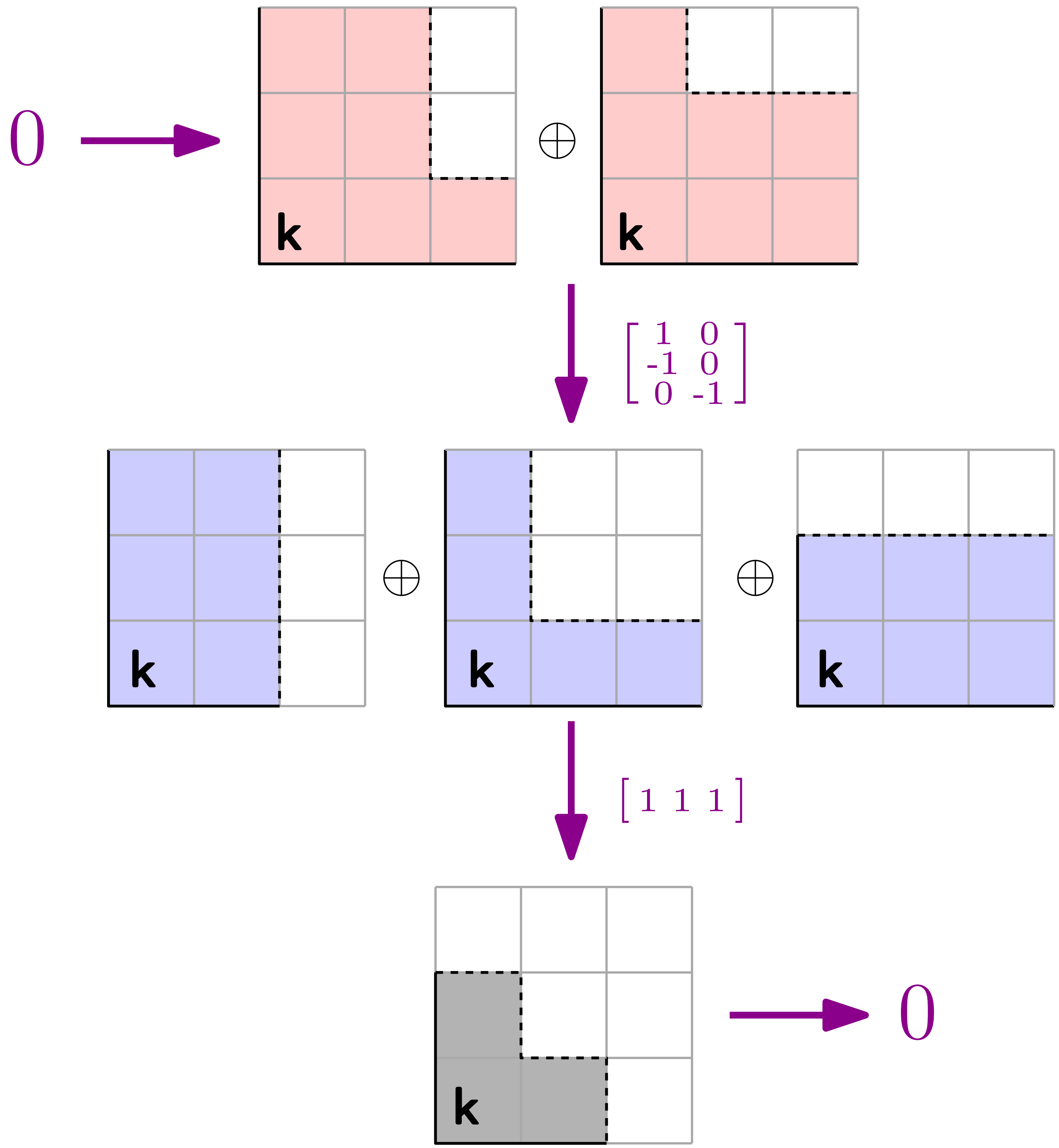
- P upper semi-lattice $\Longrightarrow$ the $\left[\boldsymbol{k}_{\langle i,j\langle}\right]$ for $i < j \in P \cup \{\infty\}$ generate $K_0(\mathcal{E}_{\mathrm{Rk}}^{\mathrm{fp}})$
- In both cases, the $\left[\boldsymbol{k}_{\langle i,j\langle}\right]$ actually form a basis, and Rk defines a monomorphism of abelian groups $K_0(\mathcal{E}) \to \mathbb{Z}^{\mathrm{Seg}(P)}$

The rank-exact structure



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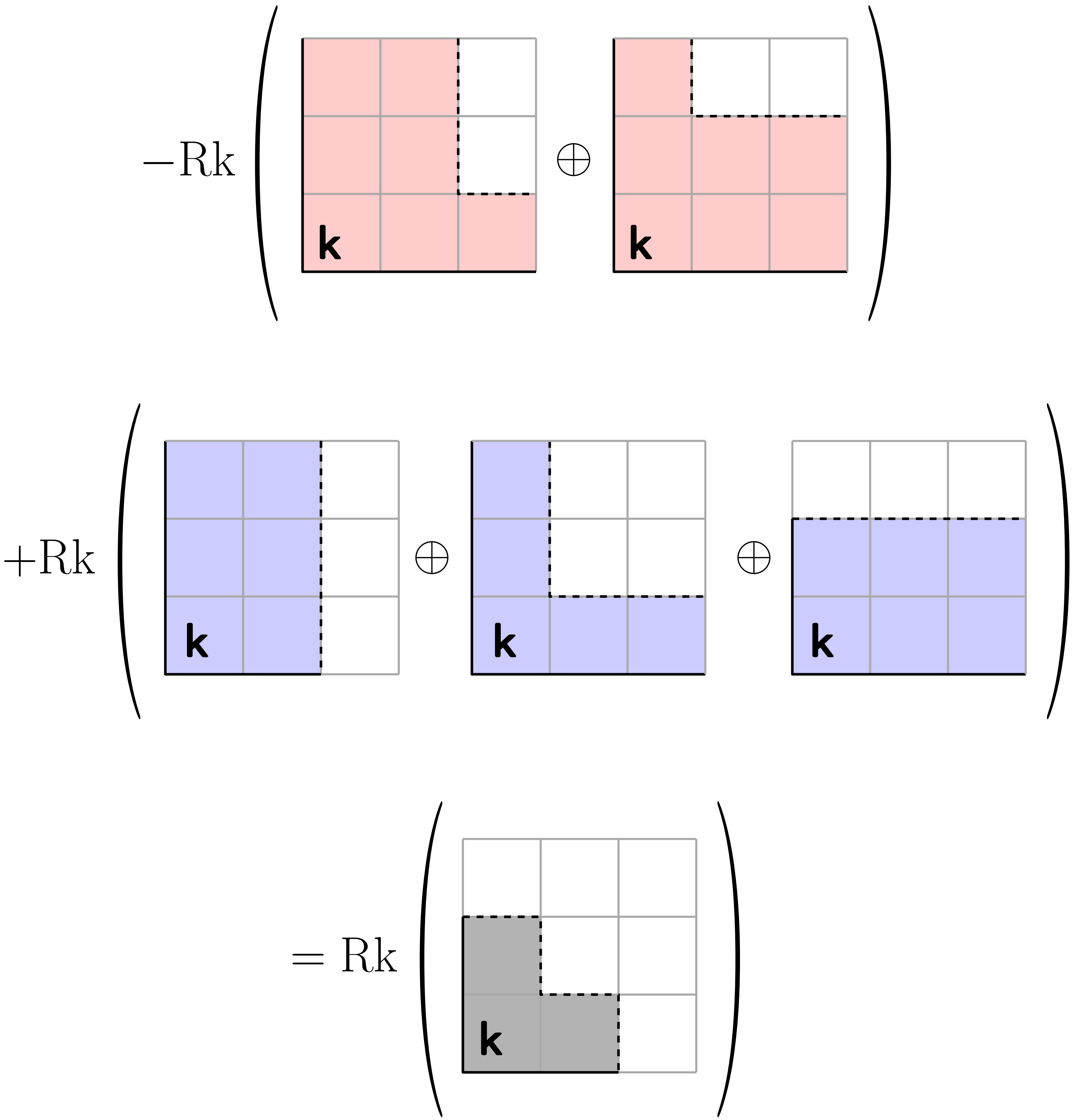

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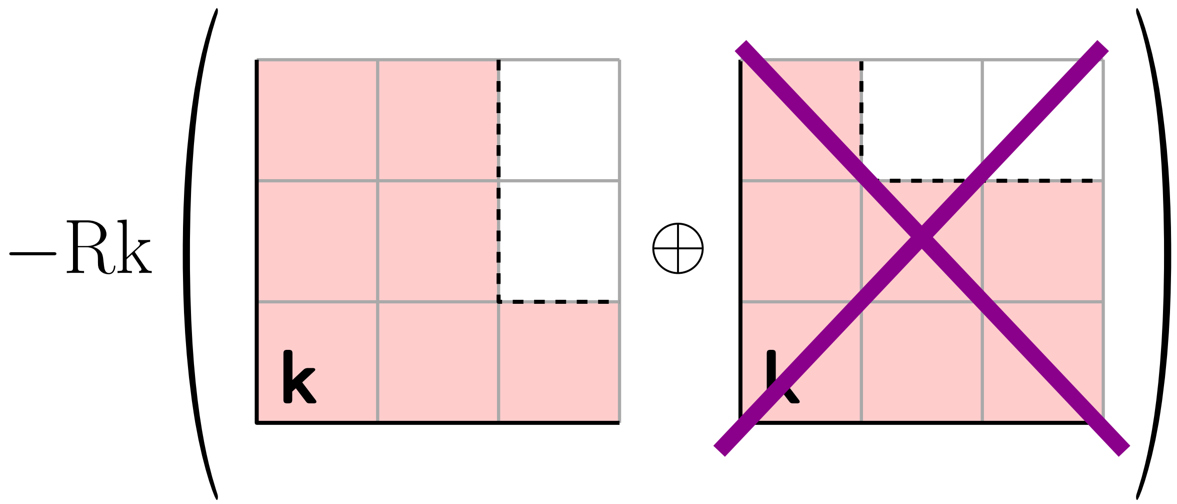
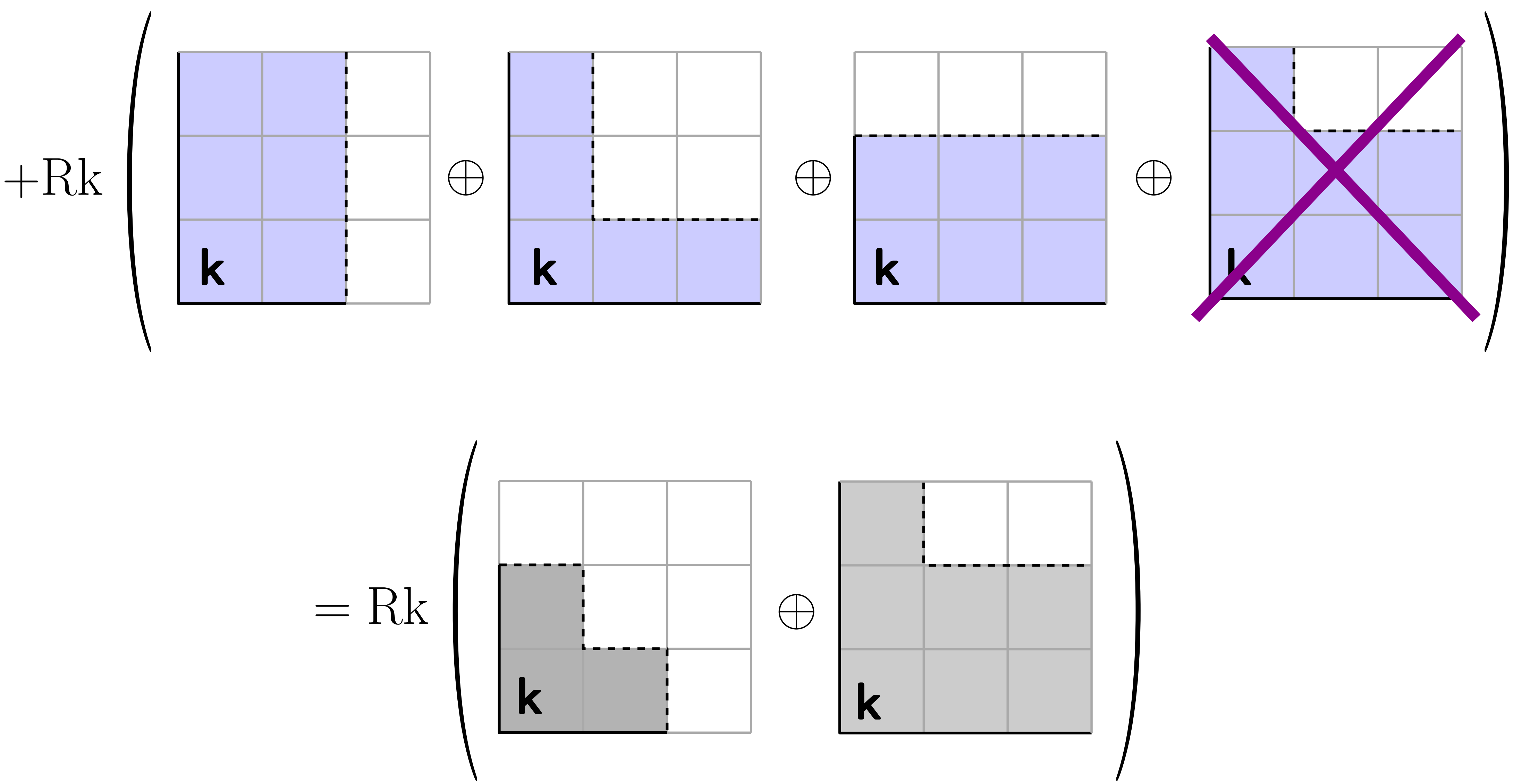
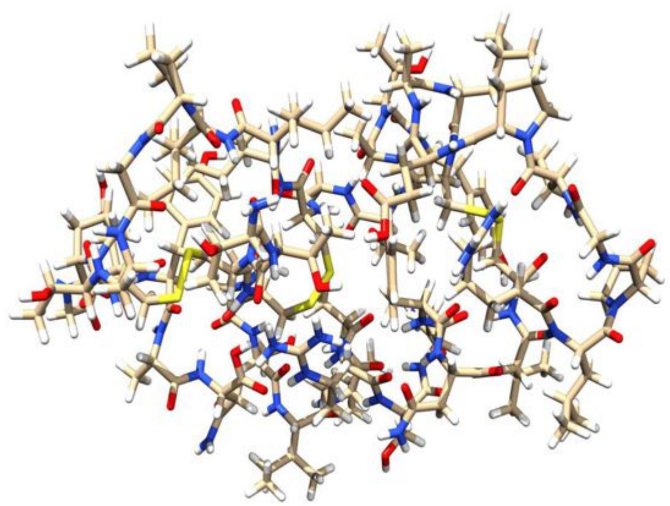


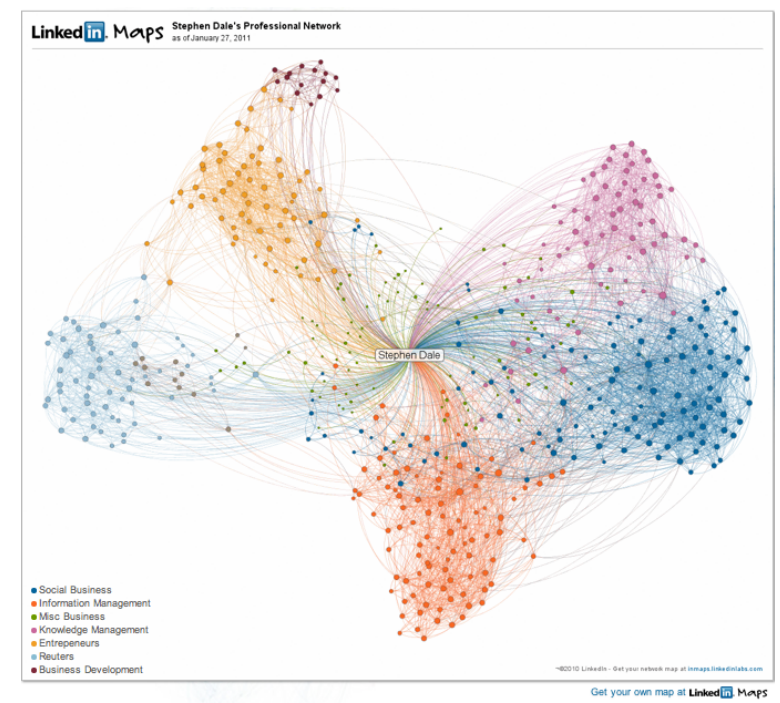
identical terms cancel out in minimal rank decomposition



Application to graph classification

- various classification tasks on molecules and social network graphs
- 5-fold evaluation (80% train, 20% test)





Dataset	COX2	DHFR	IMDB-B	IMDB-M	MUTAG	PROTEINS
1-d barcode	76.0(4.1)	70.9(3.1)	54.0(1.9)	36.3(1.1)	79.2(7.7)	65.4(2.7)
MP-Kernel	**79.9(1.8)**	**81.7(1.9)**	68.2(1.2)	46.9(2.6)	86.1(5.2)	67.5(3.1)
MP-Landscapes	79.0(3.3)	79.5(2.3)	71.2(2.0)	46.2(2.3)	84.0(6.8)	65.8(3.3)
MP-Images	77.9(2.7)	80.2(2.2)	71.1(2.1)	46.7(2.7)	85.6(7.3)	67.3(3.5)
GRIL	79.8(2.9)	77.6(2.5)	65.2(2.6)	NA	87.8(4.2)	70.9(3.1)
Rank inv.	78.2(1.7)	79.9(2.1)	73.0(4.5)	49.1(1.6)	87.2(5.8)	70.2(2.1)
Rank decomp.	78.4(0.7)	78.7(1.7)	**75.1(3.4)**	**51.1(1.3)**	**89.9(4.3)**	**73.9(1.7)**
Baseline (10-fold)	**80.1**	81.5	74.3	**52.4**	**92.1**	**76.3**

More on homological invariants for TDA

- Framework: $\dim\text{-}\mathrm{Hom}$ vs. homological invariants [Blanchette, Brüstle, Hanson]

  https://arxiv.org/abs/2112.07632

- Computation via Koszul complexes [Chacholski et al.]

  https://arxiv.org/abs/2209.05923

- Hilbert function: decomposition and stability [O., Scoccola]

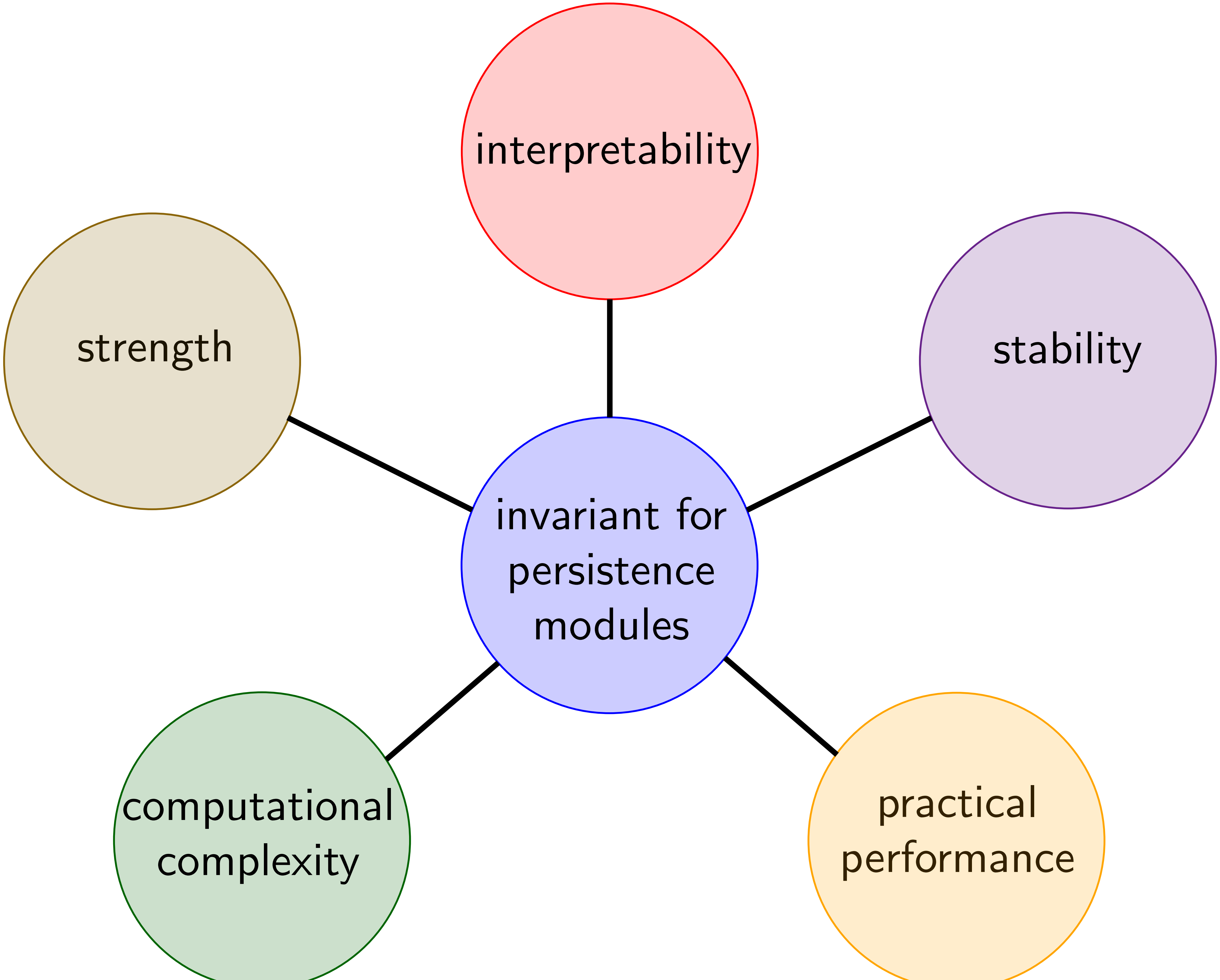
  https://arxiv.org/abs/2112.11901

- Rank invariant: decomposition and stability [Botnan, Oppermann, O., Scoccola]

  https://arxiv.org/abs/2107.06800

  https://arxiv.org/abs/2208.00300

Take-home message



Homological invariants

Definition: An *additive invariant* is a map $\alpha\colon \mathsf{Vect}_{\mathrm{fp}}^{\mathbb{R}^d} \to A$, with A an Abelian group, such that $\alpha(M \oplus N) = \alpha(M) + \alpha(N)$ for all $M, N \in \mathsf{Vect}_{\mathrm{fp}}^{\mathbb{R}^d}$.

- Typical example (**dim-Hom** invariant): given a collection $\mathcal{I}$ of intervals,

$$\alpha(-) := (\dim \mathrm{Hom}(\boldsymbol{k}_I, -))_{I \in \mathcal{I}} \in \mathbb{Z}^{\mathcal{I}}$$

($\mathcal{I} =$ positive quadrants: Hilbert function)

($\mathcal{I} =$ hooks: $\dim \ker$ invariant, or equivalently rank invariant)

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▶ Typical example (**dim-Hom** invariant): given a collection $\mathcal{I}$ of intervals,

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▶ Let $\mathcal{E}_\alpha$ be the collection of short exact sequences $0 \to L \to M \to N \to 0$ on which α is additive (use the shorthand $\mathcal{E}_{\mathcal{I}}$ when $\alpha = (\dim \mathrm{Hom}(\boldsymbol{k}_I, -))_{I \in \mathcal{I}}$)

Proposition: (follows from [Auslander, Solberg])
If $\mathcal{I}$ contains all the up-sets, then:

- $\mathcal{E}_{\mathcal{I}}$ forms an exact structure on $\mathrm{Vect}_{\mathrm{fp}}^{\mathbb{R}^d}$
- the indecomposable projectives relative to $\mathcal{E}_{\mathcal{I}}$ are the $\boldsymbol{k}_I$ for $I \in \mathcal{I}$
- $\mathcal{E}_{\mathcal{I}}$ has enough projectives
  - ▶ can do homological algebra with $\mathcal{E}_{\mathcal{I}}$

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- for every fp M that has a finite minimal $\mathcal{E}_{\mathcal{I}}$-resolution $M_n \to \cdots \to M_0 \twoheadrightarrow M$, we get a canonical decomposition:

$$(\dim \mathrm{Hom}(\boldsymbol{k}_I, M))_{I\in\mathcal{I}} = \sum_{j\in\mathbb{N}} (-1)^j \ (\dim \mathrm{Hom}(M_j, M))_{I\in\mathcal{I}}$$

$$= \sum_{j\in 2\mathbb{N}} (\dim \mathrm{Hom}\,(M_j, M))_{I\in\mathcal{I}} - \sum_{j\in 2\mathbb{N}+1} (\dim \mathrm{Hom}\,(M_j, M))_{I\in\mathcal{I}}$$

- canonical signed barcode $\left(\beta^{\mathcal{I}}_{2\mathbb{N}}(M), \beta^{\mathcal{I}}_{2\mathbb{N}+1}(M)\right)$ (with cancellable pairs)

Homological invariants

Question: size of the decompositions / length of the resolutions?

- case $\mathcal{I} = \{\text{up-sets}\}$: $\mathrm{gldim}\left(\mathsf{Vect}_{\mathrm{fp}}^{\mathbb{R}^d}\right) = d$ (Hilbert's Syzygy theorem)

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- case $\mathcal{I} = \{\text{hooks}\}$:

$$\operatorname{gldim}^{\mathcal{E}_{\mathcal{I}}}\left(\mathsf{Vect}_{\mathrm{fp}}^{\mathbb{R}^d}\right) = 2d - 2$$ [Botnan, Oppermann, O., Scoccola]

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- more general cases: relate $\mathcal{E}_\mathcal{I}$ to the usual exact structure on some alternative module category (hypothesis: $\mathbb{R}^d$ replaced by some finite poset P):

[Blanchette, Brüstle, Hanson] consider $\operatorname{mod} \operatorname{End}_{\boldsymbol{k}P}(T_\mathcal{I})^{\text{op}}$ where $T_\mathcal{I} = \bigoplus_{I \in \mathcal{I}} \boldsymbol{k}_I$

$$\operatorname{add}(\{\boldsymbol{k}_I \mid I \in \mathcal{I}\}) \xrightarrow[\text{ff}]{\operatorname{Hom}_{\boldsymbol{k}P}(T_\mathcal{I}, -)} \operatorname{proj}(\operatorname{mod} \operatorname{End}_{\boldsymbol{k}P}(T_\mathcal{I})^{\text{op}})$$

$$\Rightarrow \operatorname{gldim}^{\mathcal{E}_\mathcal{I}}(\operatorname{mod} \boldsymbol{k}P) \leq \operatorname{gldim}(\operatorname{mod} \operatorname{End}_{\boldsymbol{k}P}(T_\mathcal{I})^{\text{op}})$$